

The maximum term and the rank of an entire function.

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Introduction.

We indicate first of all certain notations which we shall use in this work. Let $f(z) = \sum_0^{\infty} a_n z^n$ be an entire function of finite order ϱ . Let $\mu(r)$ be its maximum term for $|z| = r$ and $\nu(r)$ its rank; $n(r, a)$ denotes the number of zeros of $f(z) - a$ in $|z| \leq r$. We write also $n(r, a) = n(r)$ and $M(r) = \max_{|z|=r} f(z)$.

An entire function $f(z)$ is said to have a as an exceptional value if

$$\limsup_{r \rightarrow \infty} \frac{\log n(r, a)}{\log r} = \varrho(a) < \varrho.$$

S. M. SHAH [1] has proved the following theorem.

For a canonical product of integral order $\varrho \geq 1$ and genus $p (= \varrho$ or $\varrho - 1)$

$$\limsup_{r \rightarrow \infty} \frac{n(r) \Phi(r)}{\log M(r)} = \infty$$

holds where $\Phi(x)$ is any positive continuous non-decreasing function of x such that

$$\int_A^{\infty} \frac{dx}{x \Phi(x)}$$

is convergent.

I prove below some results of the above nature for any entire function.

§ 1.

Theorem 1. (I) *If $f(z) = \sum_0^{\infty} a_n z^n$ is an entire function of order ϱ ($0 \leq \varrho < \infty$) and if $\Phi(x)$ is a function such that*

$$\log x = o(\Phi(x)),$$

then we have

$$\lim_{r \rightarrow \infty} \frac{\nu(r) \Phi(r)}{\log M(r)} = \infty.$$

(II) If the order ϱ is such that $0 < \varrho < \infty$, then

$$\limsup_{r \rightarrow \infty} \frac{\nu(r) \Phi(r)}{\log M(r)} = \infty$$

for any $\Phi(x)$ tending to infinity.

We observe that the hypothesis on $\Phi(x)$ in the first part of the above theorem allows the integral

$$\int_A^{\infty} \frac{dx}{x \Phi(x)}$$

to be divergent: for instance by taking $\Phi(x) = \log x \log \log x$.

Proof. (I). It is sufficient to prove that

$$\limsup_{r \rightarrow \infty} \frac{\log \mu(r)}{\nu(r) \Phi(r)} = 0,$$

as the result then follows since $\log \mu(r) \sim \log M(r)$. Now

$$\log \mu(r) = A + \int_{r_0}^r \frac{\nu(x)}{x} dx < \nu(r) \left[\log \left(\frac{r}{r_0} \right) \right] + O(1).$$

So

$$\log \mu(r) < 2\nu(r) \log r.$$

Hence

$$\log \mu(r) / \nu(r) \Phi(r) < 2 \log r / \Phi(r)$$

and the result follows from (1).

The proof of (II) is immediate with the help of theorem 1 of S. M. SHAH [3].

In the first part of the theorem if

$$\limsup_{r \rightarrow \infty} \frac{\log M(r)}{(\log r)^2} < \infty,$$

then $\nu(r)$ can be replaced by $n(r)$ as in such cases $\nu(r) \sim n(r)$ holds [4].

But if we do not impose any restrictions as to the nature of the function, then $\nu(r)$ cannot always be replaced by $n(r)$. For instance if $z=0$ is an exceptional value of $f(z)$, then

$$\lim_{r \rightarrow \infty} \frac{n(r) \Phi(r)}{\log M(r)} = 0,$$

because then

$$\log M(r) \sim Tr^{\varrho} \quad (0 < T < \infty)$$

and

$$n(r) = O(r^c) \quad (c < \varrho).$$

So

$$\frac{n(r) \Phi(r)}{\log M(r)} < \frac{Kr^c \Phi(r)}{\log M(r)} \sim \frac{K' \Phi(r)}{r^{\varrho-c}} \rightarrow 0$$

where we can take $\Phi(x)$ even as large as x^{δ} ($0 < \delta < \varrho - c$).

We are proving now the

Theorem 2. *If a is an exceptional value of $f(z)$ then*

$$\lim_{r \rightarrow \infty} \frac{n(r, x) \Phi(r)}{\log M(r)} = \infty \quad \text{for all } x \neq a$$

and for any $\Phi(x)$ tending to infinity.

Proof. Let $\varrho(r)$ be the LINDELÖF approximate order of $f(z)$, then

$$\frac{n(r, a)}{r^{\varrho(r)}} \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

because

$$n(r, a) = O(r^c) \quad (c < \varrho)$$

and

$$\varrho(r) \rightarrow \varrho \quad \text{as } r \rightarrow \infty.$$

Hence the ratio $\frac{n(r, x)}{r^{\varrho(r)}}$ has a positive lower bound (see [4], p. 87) for all $x \neq a$.

Now

$$\frac{n(r, x) \Phi(r)}{\log M(r)} = \frac{n(r, x)}{r^{\varrho(r)}} \frac{r^{\varrho(r)}}{\log M(r)} \Phi(r)$$

and further $\log M(r) \leq r^{\varrho(r)}$ for all $r \geq r_0$. Hence

$$\frac{n(r, x) \Phi(r)}{\log M(r)} \geq A \Phi(r) \rightarrow \infty \quad \text{as } r \rightarrow \infty$$

for any $\Phi(r)$ tending to infinity.

Theorem 3. *If $f(z)$ is a canonical product of integral order ϱ and of genus $p = \varrho$ and if $\frac{n(r) \Phi(r)}{r^{p+1}}$ has a positive lower bound, then*

$$\limsup_{r \rightarrow \infty} \frac{n(r) \Phi(r)}{\log M(r)} = \infty.$$

*Proof.*¹⁾ Since

$$\log M(r) \leq K \left[r^p \int_a^r \frac{n(x)}{x^{p+1}} dx + r^{p+1} \int_r^\infty \frac{n(x)}{x^{p+2}} dx \right]$$

it is enough to prove that no finite $C > 0$ can satisfy

$$(1.1) \quad r^p \int_a^r \frac{n(x)}{x^{p+1}} dx + r^{p+1} \int_r^\infty \frac{n(x)}{x^{p+2}} dx > n(r) \Phi(r) C$$

¹⁾ We give the proof assuming that the function has no zeros at the origin; if it has, then a slight modification of the proof gives the same result.

for all sufficiently large r , e. g. for $r \geq R$. Now since $\int \frac{n(x)}{x^{p+2}} dx$ is always convergent, let us choose R so that

$$\int_R^{\infty} \frac{n(x)}{x^{p+2}} dx < \varepsilon.$$

Let us suppose that (1.1) holds for $r = R$, then

$$(1.2) \quad R^p \int_{\alpha}^R \frac{n(x)}{x^{p+1}} dx + R^{p+1} \int_R^{\infty} \frac{n(x)}{x^{p+2}} dx > n(R) \Phi(R) C,$$

and

$$n(x) = O(x^{p+\varepsilon}).$$

So

$$\int_{\alpha}^R \frac{n(x) dx}{x^{p+1}} < k \int_{\alpha}^R x^{\varepsilon-1} dx = o(R).$$

Hence the left hand side of (1.2) $= o(R^{p+1})$. So

$$\frac{n(R) \Phi(R)}{R^{p+1}} \rightarrow 0$$

in contradiction with our hypothesis, and this completes the proof of the theorem.

§ 2.

T. VIJAYA RAGHAVAN has proved [5] that

$$M'(r) \geq \frac{M(r)}{r} \frac{\log M(r)}{\log r} \quad \text{for } r > r_0(f).$$

We prove here an analogous result for $\mu(r)$:

Theorem 4.

$$\mu'(r) \geq \frac{\mu(r)}{r} \frac{\log \mu(r)}{\log r}$$

is valid for every $r > r_0(f)$.

Proof.

$$\log \mu(r) = A + \int_{\alpha}^r \frac{v(x)}{x} dx;$$

hence

$$(2.1) \quad \frac{\mu'(r)}{\mu(r)} = \frac{v(r)}{r}$$

and

$$(2.2) \quad \log \mu(r) = \log |a_n| + n \log r \quad \text{for } R_n \leq r < R_{n+1}.$$

Since for any entire function

$$\limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}} = 0$$

holds, we have

$$(2.3) \quad |a_n| < 1 \quad \text{for } n \geq n_0.$$

Hence $\log |a_n|$ is negative and from (2.2) we get

$$(2.4) \quad \frac{\log \mu(r)}{\log r} \leq \nu(r) \quad \text{for } r \geq r_0(f).$$

Combining (2.1) and (2.4) we get the result of the theorem.

§ 3.

Theorem 5. *If $\mu(r_0) > 1$ and one of the integrals*

$$I_1 = \int_{r_0}^{\infty} \frac{\log \mu(t) dt}{t^{m+1}}$$

and

$$I_2 = \int_{r_0}^{\infty} \frac{\nu(t) dt}{t^{m+1}}$$

converges resp. diverges, then the other converges resp. diverges too.

Proof.

$$\int_{r_0}^r \frac{\nu(t) dt}{t} = \log \mu(r) - \log \mu(r_0).$$

Hence

$$\begin{aligned} & \int_{r_0}^u \frac{dr}{r^{m+1}} \int_{r_0}^r \frac{\nu(t) dt}{t} = \int_{r_0}^u [\log \mu(r) - \log \mu(r_0)] \frac{dr}{r^{m+1}} = \\ & = \left[\frac{\log \mu(r) - \log \mu(r_0)}{-mr^m} \right]_{r_0}^u + \frac{1}{m} \int_{r_0}^u \frac{\nu(r) dr}{r^{m+1}} = \\ & = \frac{\log \mu(u) - \log \mu(r_0)}{-mu^m} + \frac{1}{m} \int_{r_0}^u \frac{\nu(r) dr}{r^{m+1}} = \\ & = \int_{r_0}^u [\log \mu(r) - \log \mu(r_0)] \frac{dr}{r^{m+1}} = \\ & = \int_{r_0}^u \frac{\log \mu(r) dr}{r^{m+1}} - \log \mu(r_0) \int_{r_0}^u \frac{dr}{r^{m+1}} = \int_{r_0}^u \frac{\log \mu(r) dr}{r^{m+1}} + \frac{\log \mu(r_0)}{m} \left[\frac{1}{u^m} - \frac{1}{r_0^m} \right]. \end{aligned}$$

Thus

$$\int_{r_0}^r \frac{\log \mu(t)}{t^{m+1}} dt + \frac{\log \mu(r_0)}{m} \left[\frac{1}{r^m} - \frac{1}{r_0^m} \right] = \frac{\log \mu(r) - \log \mu(r_0)}{-m r^m} + \frac{1}{m} \int_{r_0}^r \frac{r(t) dt}{t^{m+1}}$$

and so

$$(3.1) \quad m \int_{r_0}^r \frac{\log \mu(t) dt}{t^{m+1}} - \frac{\log \mu(r_0)}{r_0^m} + \frac{\log \mu(r)}{r^m} = \int_{r_0}^r \frac{r(t)}{t^{m+1}} dt.$$

We suppose now that I_1 is convergent; then

$$\varepsilon > \int_r^{2r} \frac{\log \mu(t) dt}{t^{m+1}} > \frac{\log \mu(r)}{m r^m} \left[1 - \frac{1}{2^m} \right] \text{ for every } \varepsilon > 0.$$

Consequently

$$\frac{\log \mu(r)}{r^m} \rightarrow 0.$$

Hence

$$m I_1 + K = I_2 \quad \left(K = -\frac{\log \mu(r_0)}{r_0^m} \right)$$

holds and this implies the convergence of I_2 .

Similarly if I_2 is convergent, then

$$m \int_{r_0}^r \frac{\log \mu(t) dt}{t^{m+1}} + \frac{\log \mu(r)}{r^m} < K'$$

and as

$$\int_{r_0}^r \frac{\log \mu(t) dt}{t^{m+1}} > \log \mu(r_0) \frac{1}{m} \left[\frac{1}{r_0^m} - \frac{1}{r^m} \right] > 0$$

both terms on the left hand side are positive and this secures the convergence of I_1 .

From the convergence of $\int_{r_0}^{\infty} \frac{r(t)}{t^{m+1}} dt$ we can also deduce that $\log \mu(r) = o(r^m)$.

Further we get from (3.1) that

$$\int_{r_0}^r \frac{r(t) dt}{t^{m+1}} > \int_{r_0}^r \frac{\log \mu(t) dt}{t^{m+1}}.$$

Hence the divergence of I_2 follows from the divergence of I_1 .

Now let I_2 be divergent. Then I_1 will also be divergent for if it were convergent, then by the results established above I_2 will also be convergent in contradiction with our hypothesis.

§ 4.

In [6] I have proved that for $0 < r < R$

$$\frac{M(R)}{m(r)} \cong \left(\frac{R}{r}\right)^{n(r)}$$

where

$$m(r) = \min_{|z|=r} |f(z)|$$

and as usual

$$M(R) = \max_{|z|=R} |f(z)|.$$

Here we shall prove further

Theorem 6. *If $f(z)$ is an entire function having no zeros in the unit circle, then*

$$\frac{M(R)}{m(r)} \cong \left(\frac{R}{r}\right)^{\frac{N(R)}{\log R}} \quad (0 < r < R)$$

where

$$N(R) = \int_0^R \frac{n(t)}{t} dt.$$

Proof.

$$(4.1) \quad \begin{aligned} N(R) - N(r) &= \int_r^R \frac{n(t)}{t} dt = \int_0^R \frac{n(t)}{t} dt - \int_0^r \frac{n(t)}{t} dt \leq \\ &\leq \log M(R) - \log m(r) \end{aligned}$$

by JENSEN'S theorem. $N(x)$ is an increasing convex function of $\log x$. If we draw the graph of the function $N(x)$, it will pass through the origin. Let O be the origin and $A(\log R, N(R))$, $B(\log r, N(r))$, be two points on the graph. Then the slope of OA is greater than the slope of OB . Hence

$$\frac{N(R)}{\log R} \geq \frac{N(r)}{\log r}$$

and it follows that

$$\frac{N(R) - N(r)}{\log R - \log r} \geq \frac{N(R)}{\log R}.$$

Thus (4.1) gives

$$\frac{N(R)}{\log R} \leq \frac{\log M(R) - \log m(r)}{\log R - \log r}$$

from which the result of the theorem follows.

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References.

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