Abelian groups in which every serving subgroup is a direct summand.

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§ 1. Introduction.

It is an interesting problem to characterize all groups in which all subgroups of a given type have certain special properties. Such problems have been discussed recently by the second-named author[1]) (all subgroups are direct summands) as well as by the second- and third-named authors[2]) who have determined those abelian groups every multiple of which is a direct summand and have given a rather full description of abelian groups every endomorphic image of which is a direct summand. Perhaps it may be of some interest to discuss an entirely analogous problem which arises if we replace the term "multiple" by "serving subgroup":[3]) *to characterize all abelian groups G with*

Property P. *Every serving subgroup of G is a direct summand of G.*

Let us observe that since the direct summands of G are obviously serving subgroups of G, our problem may also be considered to consist in finding all abelian groups in which the two notions: "serving subgroup" and "direct summand" coincide.

In what follows we shall completely solve the stated problem. In §§ 3—5 we shall give a characterization of all torsion, torsion free and mixed groups of this type. Our result may be formulated as follows:

A necessary and sufficient condition that an abelian group G have Property P is that G be representable as a direct sum

$$G = A + B$$

satisfying the following conditions:

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[1]) See KERTÉSZ [3]. — The numbers in brackets refer to the Bibliography given at the end of this paper.

[2]) See KERTÉSZ and SZELE [5].

[3]) For the terminology and notation we refer to § 2.

a) *A is an algebraically closed group, i. e. the direct sum of quasicyclic groups and groups isomorphic to the additive group $\mathfrak{R}$ of all rational numbers;*

b) *B is either the direct sum of cyclic p-groups such that, for each fixed prime p, the orders of the cyclic p-groups are bounded, or the direct sum of a finite number of groups which are isomorphic to the same proper subgroup of $\mathfrak{R}$. If A is not a torsion group, then for B only the the second alternative is possible.*

The final section of this paper, § 6, is devoted to determining all abelian groups in which every subgroup is serving. These groups coincide with the elementary abelian groups just as in the case of the problem discussed in [3].[4])

§ 2. Preliminaries.

By a *group* we shall mean throughout a non-trivial abelian group G written additively. If every element of G has a finite order then G is called a *torsion* group. Recall that each torsion group may be represented in a unique way as the direct sum of *p-groups* (called its *p-components*), these being groups in which the orders of the elements are powers of one and the same prime p. An *elementary* group is a torsion group whose elements have square free orders. If all non-zero elements of G are of infinite order, then G is a *torsion free* group, while a group which is neither a torsion nor a torsion free group is said to be a *mixed* group.

The most important special groups which are needed below are as follows. Cyclic groups of order p^n where p is a natural prime and n a natural integer (notation: $\mathfrak{Z}(p^n)$); *quasicyclic* groups or groups of type p^∞ (notation: $\mathfrak{Z}(p^\infty)$) which are isomorphic to the factorgroup of the additive group of all rational numbers whose denominator is a power of a prime p, modulo the subgroup of all integers; finally, *rational groups,* i. e. subgroups of the additive group $\mathfrak{R}$ of all rational numbers.

A subset $S=(a_\nu)$ of G, not containing 0, is called *independent* if for any finite subset $a_{\nu_1}, \dots, a_{\nu_k}$ of S a relation

$$n_1 a_{\nu_1} + \cdots + n_k a_{\nu_k} = 0 \qquad (n_i \text{ integers})$$

implies

$$n_1 a_{\nu_1} = \cdots = n_k a_{\nu_k} = 0.$$

By the *rank* of G we mean the cardinal number of a maximal independent system of G, containing but elements of infinite order. For example, the subgroups of $\mathfrak{R}$ are of rank 1. The converse for torsion free groups is also true: a torsion free group of rank 1 is isomorphic to some subgroup

[4]) It is to be emphasized that we assume commutativity what has not been done in [3].

R of $\mathfrak{R}$. We shall need the following simple criterion[5]): two rational groups A and B are *not* isomorphic if and only if there exists an infinite set of prime powers, $\mathfrak{P}$, which annihilates exactly one of A and B, i. e. either $\mathfrak{P}A \neq 0$ and $\mathfrak{P}B = 0$ or $\mathfrak{P}A = 0$ and $\mathfrak{P}B \neq 0$, where $\mathfrak{P}X$ denotes the intersection of all sets $p^s X$, p^s running over all elements of $\mathfrak{P}$.

If the equation $nx = a$ has a solution $x \in G$ for each $a \in G$ and all rational integers n, then G is called an *algebraically closed* (or *complete)* group. (An obviously equivalent definition is that $nG = G$ for all non-zero integers n.) It is well known[6]) that an algebraically closed group is isomorphic to a direct sum of quasicyclic groups and of groups $\mathfrak{R}$. If H is an algebraically closed subgroup of G, then H is necessarily a direct summand of G, i. e. G has a direct decomposition $G = H + F$ for some subgroup F of G.[7]) The union C of all algebraically closed subgroups of G is again algebraically closed and so we have $G = C + G'$ where G' has no algebraically closed subgroup other than 0. Such a group G' is usually called *reduced*.

For a non-zero element a of order p^n the maximal non-negative integer k for which the equation $p^k x = a$ is solvable in G is said to be the *height* of a. If there is no maximal k with this property, then a is of infinite height.

Let H be a subgroup of G. If for each $a \in H$, the solvability of $nx = a$ in G implies the solvability in H, then H is said to be a *serving* subgroup of G. (For p-groups it is clearly enough to consider the mentioned equation only for $n = p^r$.) An equivalent definition is that each coset of H contains an element whose order is the same as the order of this coset in the factorgroup G/H. It is evident that if K is serving in H and H is serving in G, then K is a serving subgroup of G.

A subgroup B of a p-group G is termed a *basic* subgroup[8]) of G if (i) B is the direct sum of cyclic groups, (ii) B is a serving subgroup of G, (iii) the factorgroup G/B is an algebraically closed group. By an important theorem of L. Kulikov,[8]) each p-group contains a basic subgroup.

The following notation will be used. The sign $+$ or Σ will denote the (discrete) direct sum of subgroups. For any non-void subset K of G, $\{K\}$ is used to denote the subgroup of G generated by the elements of K. If G is torsion free then by $\{K\}$ we shall denote the least serving subgroup in G which contains K. ($\{K\}_*$ is uniquely determined since G was supposed to be torsion free; it is obvious that this subgroup of G consists of all those elements x of G for which $nx \in K$ holds with a suitable non-zero integer n.)

Finally, we prove a simple lemma which will prove to be very useful in our investigations.

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[5]) A full characterization of the rational groups may be found, for example, in Baer [2] or in Rédei and Szele [9].

[6]) See e. g. Szele [10].

[7]) Baer [1].

[8]) Kulikov [7].

Lemma 1. *If a group G has Property P then each direct summand H of G has again Property P.*

Let K be a serving subgroup of the direct summand H of G. Then K is a serving subgroup of G and therefore G has a direct decomposition

$$G = K + L \tag{1}$$

with some subgroup L of G. Since K is a subgroup of H, (1) implies the existence of a direct decomposition

$$H = K + L'$$

for some subgroup L' of H, q. e. d.[9])

§ 3. Torsion groups with Property P.

The first step in our examinations is the discussion of p-groups having Property P. We shall prove the following theorem which, together with a recent result of one of the authors,[10]) will show that these groups are identical with the p-groups in which the heights of the elements of finite height are bounded.

Theorem 1. *A p-group G has Property P if and only if it is the direct sum of cyclic and quasicyclic groups and the cyclic summands are of bounded order, i. e., G has the form*

$$G = \sum_{\nu} \mathfrak{Z}_{\nu}(p^n) \quad \textit{where } n = 1, 2, \ldots, m \textit{ or } \infty \tag{2}$$

for some fixed integer m.

For the proof of the necessity let us consider a basic subgroup B of the group G with Property P. By definition, B is a serving subgroup and therefore we have

$$G = B + C$$

where, again by the definition of B, C is an algebraically closed group and hence C may be represented as the direct sum of quasicyclic groups. In order to complete the necessity part of the proof, we have still to show that the direct summands of B are of bounded order.

Assume, on the contrary, that the elements of B are not of bounded order and let $A = \{a_1\} + \{a_2\} + \ldots$ be an infinite direct summand of B (and

[9]) Let us remark that, in general, Property P is not hereditary under homomorphic mappings. Indeed, Theorem 3 will imply that the factorgroup G/H of a group $G = \{a\} + \{b\}$ (where a and b are of infinite order) modulo the subgroup $H = \{pa\}$ does not possess Property P. Nevertheless, if H is a serving subgroup of G, then G/H has again Property P, but this tells us nothing new than Lemma 1.

[10]) See KERTÉSZ [4]: A p-group G in which the heights of the elements of finite height are bounded has a decomposition (2).

hence also of G) such that $1 < p^{n_1} < p^{n_2} < \dots$ where $p^{n_k} = O(a_k)$.[11]) We show that the subgroup

$$D = \{a_1 - p^{n_2-n_1}a_2,\ a_2 - p^{n_3-n_2}a_3, \dots\}$$

is a serving subgroup of A not containing a_1. In fact, a relation

$$p^r(h_1a_1 + \dots + h_ka_k) = m_1(a_1 - p^{n_2-n_1}a_2) + \dots + m_s(a_s - p^{n_{s+1}-n_s}a_{s+1})$$

(h_i and m_j are integers) implies, by the independence of the a_i, that all of $m_1, \dots, m_s$ are divisible by p^r, establishing the serving character of D, while the impossibility of

$$a_1 = m_1(a_1 - p^{n_2-n_1}a_2) + \dots + m_s(a_s - p^{n_{s+1}-n_s}a_{s+1})$$

follows, by the same reason, in view of the congruences

$$m_s \equiv 0 \pmod{p^{n_s}}\ , \dots,\quad m_1 \equiv 0 \pmod{p^{n_1}}.$$

Now, from Lemma 1 we conclude

$$A = D + E$$

and hence (A being a direct summand of B) $B = D + F$ for some subgroup F of B. But this implies

$$A/D \subseteq B/D \simeq F,$$

i. e. F contains a subgroup $\mathfrak{Z}(p^\infty)$, contrary to the fact that B is reduced. Hence the stated condition is necessary.

In order to prove its sufficiency, let us suppose that G is a group with a decomposition (2). Let C denote the maximal algebraically closed subgroup of G, i. e. the direct sum of the quasicyclic direct summands, H an arbitrary serving subgroup of G and H_1 the intersection $H \cap C$. On account of $H_1 \subseteq C$, each equation of the form

$$p^{n+m}x = h \in H_1$$

must be solvable for some x in G, and therefore also for some x' in H. Then $y = p^m x' \in H$ solves the equation

$$p^n y = h \in H_1$$

and by the choice of m we have $y \in C$ whence $y \in H_1$. Consequently, H_1 is an algebraically closed group. This result leads us to a decomposition

$$H = H_1 + H_2.$$

H_2 as a serving subgroup of the serving subgroup H is serving in G and since the orders of the elements in H_2 are bounded, by a theorem of KULIKOV[12]) we obtain that the reduced group H_2 is a direct summand of G. Finally, H_1 as an algebraically closed group is a direct summand of every group containing it, therefore we arrive at the result that $H = H_1 + H_2$ is a direct summand of G. This completes the proof of the theorem.

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[11]) $O(a)$ denotes the order of the group element a.

[12]) KULIKOV [6]: If in a p-group G, H is a serving subgroup in which the orders of the elements are bounded, then H is a direct summand of G.

Now it is easy to pass from p-groups to arbitrary torsion groups.

Theorem 1a. *An abelian torsion group G possesses Property P if and only if it is the direct sum of cyclic p-groups and quasicyclic groups such that, for each fixed prime p, the orders p^n of the cyclic direct summands $\mathfrak{Z}(p^n)$ are bounded.*

The assertion of Theorem 1a is an obvious consequence of Theorem 1 and the following simple observation: *If G is a torsion group and G_p is a p-component of G, then G has Property P if and only if each G_p has Property P.* In fact, if G has Property P, then by Lemma 1 each G_p must again have this property. Conversely, if each p-component G_p enjoys Property P and $H=\sum_p H_p$ is a serving subgroup of G (where H_p denotes the p-component of H), then H_p is a serving subgroup of G_p and hence by hypothesis we obtain $G_p=H_p+K_p$ for some p-group K_p. This implies at once

$$G=\sum_p G_p=\sum_p (H_p+K_p)=\sum_p H_p+\sum_p K_p=H+K$$

with $K=\sum_p K_p$, q. e. d.

§ 4. Torsion free groups with Property P.[13])

At first we shall characterize those torsion free groups with Property P which are reduced, i. e. do not contain subgroups isomorphic to the additive group $\mathfrak{R}$ of all rational numbers. As result we obtain the following

Theorem 2. *A torsion free reduced abelian group G has Property P if and only if G has a direct decomposition*

$$G=R_1+R_2+\cdots+R_r$$

with a finite number of components where R_i are isomorphic to the same proper subgroup of the additive group of all rational numbers.

Let G be a torsion free reduced group with Property P. First of all we show that G has a finite rank. For, let us assume the contrary, i. e., G contains an infinite independent system $a_1, a_2, \ldots$. We form the serving subgroup H of G defined as

$$H=\{a_1-2a_2, a_2-3a_3, \ldots, a_k-(k+1)a_{k+1}, \ldots\}_*,$$

in other words, H consists of all those elements $x\in G$ for which an equation

$$nx=n_1(a_1-2a_2)+\cdots+n_k(a_k-(k+1)a_{k+1}) \qquad (n_k\neq 0) \tag{3}$$

[13]) In a letter to T. Szele, Professor A. G. Kurosh has informed us that his pupil A. P. Mishina has deduced our results in § 4 from certain theorems of R. Baer in [2]. It seems to us that it might be some interest in our present proof which does not appeal to deep results, is more direct and rather elementary.

holds with suitable integers $n, n_1, \ldots, n_k$. Since, by the independence of the a_i, (3) with x replaced by a_1 implies $n_k = 0$, we see that $a_1 \notin H$ and hence H is a proper subgroup of G. Therefore, G/H is a torsion free group containing a non-zero algebraically closed subgroup generated by the cosets containing $a_1, a_2, \ldots$, consequently, in the reduced group G the serving subgroup H can not be a direct summand. Hence G is of finite rank r, indeed.

Now let us consider a non-zero element a in G and the serving subgroup $G_1 = \{a\}_*$ of rank 1. By hypothesis we have

$$G = G_1 + H_1$$

for some subgroup H_1 of G where, evidently, H_1 is of rank $r-1$. In view of Lemma 1, using the same argument for H_1 in the place of G etc., we finally conclude

$$G = G_1 + G_2 + \cdots + G_r \tag{4}$$

where G_t are rational groups.

Next we show that all components G_i in (4) are isomorphic to one and the same rational group R. For definiteness let us assume that G_1 and G_2 are not isomorphic. Considering that $H = G_1 + G_2$ must have Property P if the same is true for G, it is enough to consider only H and prove that our last assumption leads to a contradiction. Suppose G_1 and G_2 are not isomorphic and let a, b be arbitrary non-zero elements of G_1, G_2. By a remark in § 2, there exists a set of prime powers, $\mathfrak{P}$, such that e. g.

$$\mathfrak{P}G_1 \neq 0, \quad \mathfrak{P}G_2 = 0.$$

Since G_1 is not isomorphic to $\mathfrak{R}$, there is an integer q for which the equation $qx = a$ has no solution in G_1. We show that the serving subgroup $H_1 = \{a + qb\}_*$ can not be a direct summand of G. First we observe that obviously $\mathfrak{P}H_1 = 0$ holds. Further, if we had $H = H_1 + H_2$ for some subgroup H_2, of rank 1, of H, then we should have

$$\mathfrak{P}H_2 = \mathfrak{P}H_1 + \mathfrak{P}H_2 = \mathfrak{P}H = \mathfrak{P}G_1 + \mathfrak{P}G_2 = \mathfrak{P}G_1 \neq 0$$

whence H_2 contains an element of the form $ta \neq 0$ (t a rational number). Considering that H_2 is of rank 1, it follows that any element of H_2 has the same form. Therefore, $H = H_1 + H_2$ implies

$$b = s(a + qb) + ta$$

with rational numbers s, t. Hence we conclude $-t = s = \dfrac{1}{q}$, $\quad q(-ta) = a$, in contradiction to the choice of q. This establishes the isomorphism of G_1 and G_2.

From what has been said it follows that $G_1, G_2, \ldots, G_r$ are isomorphic to the same rational group R and if a_i $(i = 1, \ldots, r)$ denotes the element in G_i which corresponds to a fixed element of R, say, to 1, then we may write G in the following form:

$$G = Ra_1 + Ra_2 + \cdots + Ra_r. \tag{5}$$

In the proof of the sufficiency we shall need the following lemma which may be considered as a slight generalization of a lemma due to R. RADO.[14])

Lemma 2. *If* $H = Ra_1 + Ra_2 + \cdots + Ra_k$ *and* $n_1, n_2, \ldots, n_k$ *are arbitrary rational integers such that* $(n_1, n_2, \ldots, n_k) = 1$, *then* H *may be written in the form*

$$H = Rb_1 + Rb_2 + \cdots + Rb_k$$

with $b_1 = n_1 a_1 + n_2 a_2 + \cdots + n_k a_k$.

The statement is obvious if $N = |n_1| + |n_2| + \cdots + |n_k| = 1$. We assume $N > 1$ and use an induction with respect to N. $N > 1$ and $(n_1, \ldots, n_k) = 1$ imply that at least two of the n_i do not vanish, say, $|n_1| \geqq |n_2| > 0$. Then we have either $|n_1 + n_2| < |n_1|$ or $|n_1 - n_2| < |n_1|$ whence

$$|n_1 \pm n_2| + |n_2| + \cdots + |n_k| < N$$

for one of the two signs. $(n_1 \pm n_2, n_2, \ldots, n_k) = 1$ and the induction hypothesis imply

$$H = Ra_1 + Ra_2 + \cdots + Ra_k = Ra_1 + R(a_2 \mp a_1) + Ra_3 + \cdots + Ra_k = \\ = Rb_1 + Rb_2 + \cdots + Rb_k$$

with

$$b_1 = (n_1 \pm n_2)a_1 + n_2(a_2 \mp a_1) + n_3 a_3 + \cdots + n_k a_k = n_1 a_1 + n_2 a_2 + \cdots + n_k a_k$$

completing the proof of the lemma.

Turning our attention to the proof of the sufficiency of the condition in Theorem 2, let us denote by H an arbitrary (non-zero) serving subgroup of a group G having the form (5). Each non-zero element b of H has a unique representation

$$b = n_1 a_1 + \cdots + n_k a_k \qquad (n_i \text{ rational}, \ n_k \neq 0).$$

Let now $b = b_k$ be an element in H for which k is as maximal as possible, further $n_1, \ldots, n_k$ are all rational integers and $|n_k|$ is minimal. Since H is a serving subgroup of G, we must have then $(n_1, \ldots, n_k) = 1$. Hence we can apply Lemma 2 to conclude that there exists a direct decomposition of the form

$$G = Rb_1 + Rb_2 + \cdots + Rb_k + Ra_{k+1} + \cdots + Ra_r.$$

Now each non-zero element b of H has the form

$$b = m_1 b_1 + \cdots + m_l b_l \qquad (m_j \text{ rational}, \ m_l \neq 0, \ l \leqq k).$$

Among the elements with $l < k$ we choose a b with a maximal l where $m_1, \ldots, m_l$ are integers and $|m_l|$ is minimal. Then we have $(m_1, \ldots, m_l) = 1$ and apply again Lemma 2 and so on. Finally, we arrive at a direct decomposition

$$G = Rc_1 + Rc_2 + \cdots + Rc_r$$

such that, by a suitable choice of notation,

$$H \subseteq Rc_1 + \cdots + Rc_s \qquad (c_1, \ldots, c_s \in H).$$

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[14]) RADO [8] or SZELE [11]. The present proof follows exactly the same lines.

Taking into account that H is a serving subgroup of G, we get $Rc_i \subseteq H$ for $i = 1, \ldots, s$, and hence we obtain

$$H = Rc_1 + \cdots + Rc_s.$$

This implies at once

$$G = H + K$$

where $K = Rc_{s+1} + \cdots + Rc_r$, q. e. d.

In order to give a full description of the torsion free groups with Property P, we prove a simple lemma.

Lemma 3. *Let C be the maximal algebraically closed subgroup of a torsion free group G and $G = C + G'$. Then G has Property P if and only if the reduced group G' has the same property.*

The necessity follows immediately from Lemma 1. In order to prove its sufficiency, let us suppose that G' has Property P. If H is a serving subgroup of G, then the equation $nx = h$ ($h \in H \cap C$) being solvable in G, its unique solution lies in C and in H, i. e. in $H \cap C$. This implies that the maximal algebraically closed subgroup of H coincides with $H \cap C$, and therefore there exists a decomposition

$$H = (H \cap C) + H'$$

where — without loss of generality — we may suppose $H' \subseteq G'$. By hypothesis we have $G' = H' + K$ and, since $C = (H \cap C) + L$,

$$G = C + G' = (H \cap C) + L + H' + K = H + (K + L)$$

which establishes the statement.

Recalling that a torsion free algebraically closed group is the direct sum of groups isomorphic to $\mathfrak{R}$, Theorem 2 and Lemma 3 together imply the desired result:

Theorem 2a. *A torsion free abelian group G possesses Property P if and only if it is of the form*

$$G = \sum_{\nu} R_{\nu}$$

where the groups R_{ν} are isomorphic to some subgroups of the additive group $\mathfrak{R}$ of all rational numbers such that those R_{ν} which are isomorphic to proper subgroups of $\mathfrak{R}$ are finite in number and isomorphic to each other.

§ 5. Mixed groups with Property P.

For mixed groups of Property P we have the following

Theorem 3. *A mixed group G has Property P if and only if it is of the form*

$$G = \sum_{\nu} G_{\nu} \tag{6}$$

where each G_ν is isomorphic to a quasicyclic group or to a rational group such that the proper subgroups of $\Re$ are in a finite number and are isomorphic to each other.

Assume G is a mixed group with Property P and T is its maximal torsion subgroup. Then T is serving in G and so we have a direct decomposition $G = T + F$ where F is an adequate torsion free subgroup of G. Lemma 1 implies that both T and F have Property P, consequently, Theorems 1a and 2a imply a decomposition (6) where G_ν are subgroups of $\mathfrak{Z}(p^\infty)$ or $\Re$ satisfying the conditions of Theorems 1a and 2a. What remains to be verified is that no G_ν is a finite cyclic group. For the proof let us suppose that $G_1 = \{a\} \simeq \mathfrak{Z}(p^n)$ $(n < \infty)$ and $G_2 \simeq R$ where R is a rational group. The existence of such groups G_1 and G_2 follows from the assumption according to which G is a mixed group. We consider $H = G_1 + G_2$ and denote by H_1 the serving subgroup of H generated by $a + pb$ where b is an arbitrary non-zero element of G_2. By virtue of the fact that H has again Property P we infer $H = H_1 + H_2$ where H_2 must be a torsion group, considering that H is of rank 1 and the rank is an invariant of the group. But the equation $px = a + pb$ has no solution in H and therefore b can not belong to $H_1 + H_2$, a contradiction. This establishes the necessity of our condition.

Conversely, let the mixed group G have a decomposition (6) with the mentioned properties and H a serving subgroup of G. If T_1 is the maximal torsion subgroup of H, then T_1 is serving in G and hence T_1 is an algebraically closed group. Therefore we obtain a direct decomposition $H = T_1 + F_1$ for a certain torsion free subgroup F_1 of H. Taking $T \cap F_1 = 0$ into account, by usual arguments we conclude

$$G = T + F \qquad \text{with } F_1 \subseteq F.$$

With regard to the facts that $F \simeq G/T$ is a group covered by Theorem 2a and F_1 is a serving subgroup of F, we get $F = F_1 + F_2$ whence

$$G = T + F = (T_1 + T_2) + (F_1 + F_2) = H + (T_2 + F_2).$$

Hereby Theorem 3 has been proved completely.

Theorems 1a, 2a and 3 together settle our stated problem.

§ 6. Abelian groups in which every subgroup is serving.

Our problem considered so far suggests an other problem closely related to it: *Which are the abelian groups every subgroup of which is serving?* A complete answer to this question is contained in the following theorem.

Theorem 4. *An abelian group G has the property that every subgroup of it is serving if and only if G is an elementary abelian group.*

Let G be an abelian group in which every subgroup is serving. Then G can not contain elements whose order is not a square free number, for in the contrary case G would also contain an element g of infinite order or of order p^2 for some prime p. This is, however, impossible considering that in this case $\{pg\}$ is by no means a serving subgroup in G. Hence G is an elementary group, in fact.

Conversely, let G be an elementary group and H a subgroup of G. Then assuming $nx = h \in H$ has a solution x in G, we decompose $h = h_1 + \cdots + h_s$ such that $h_i \in H$ and the orders of h_i are different primes p_i. In view of the existence of a solution $x \in G$ of the above equation, it is obvious that no p_i divides n whence it follows that there are multiplies $h'_i = m_i h_i$ with $n h'_i = h_i$. Then $x = h'_1 + \cdots + h'_s$ is a desired solution.

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