

Some saddle-points in $A \otimes \bar{A}$.

By DAVID ELLIS in Gainesville, Florida.¹⁾

§ 1. Lattice saddle-points.

Let I be a complete lattice²⁾ with partial ordering $\leq$. Let p and q be two properties each of which is relevant to the members of I (that is: dichotomous in I). We call such a property a *lower property* (*upper property*) in I provided the set of all elements of I which have the property is non-null and contains all predecessors (successors) of any of its elements. If p is a lower property in I and q is an upper property in I , we say that an element of I , with respect to p and q , is a saddle-point if it has both properties and is maximal with respect to having property p and minimal with respect to having property q . If there is at least one element of I having both properties and if every such element is a saddle-point, we say that p and q are *saddle-point properties*. Clearly, this amounts to saying that the set of elements having property p and the set of elements having property q intersect in a non-null, totally unordered subset of I .

§ 2. The topolattice.

Let S be an infinite set. The topolattice³⁾ A of S is the set of all T_1 topologies definable in S with the partial ordering: $\alpha \leq \beta$ if and only if every subset of S open under β is open under α . As is well known⁴⁾, A is a complete lattice.

We denote by $\bar{A}$ the lattice obtained by dualizing the ordering of A . We denote by $A \otimes \bar{A}$ the direct product (cardinal product; cf.²⁾) of the

¹⁾ Presented to the American Mathematical Society; Christmas, 1953.

²⁾ GARRETT BIRKHOFF, *Lattice theory*. Revised ed. (American Mathematical Society, New York, 1948.)

³⁾ ROBERT BAGLEY and DAVID ELLIS, On the topolattice and permutation group of an infinite set. (In preparation.)

⁴⁾ R. VAIDYANATHASWAMI, *Treatise on Set Topology*. (Indian Mathematical Society, Madras, 1947.)

lattices A and $\bar{A}$. As is well known⁴⁾ the property of being T_2 (HAUSDORFF) is a lower property in A and the property of being bicomact is an upper property in A . In fact, we have the

Theorem (VAIDYANATHASWAMI). *The properties of being T_2 and of being bicomact are saddle-point properties in A .*

To actually obtain a class of these saddle-points other than familiar examples such as the Euclidean topology of a closed interval of real numbers we may take $s \in S$ and let τ_s be that topology in S in which all sets are open except those which have s as a member and whose complements are infinite.

§ 3. Permutation saddle-points.

Let f be any permutation of the set S . We consider the properties for (σ, τ) in $A \otimes \bar{A}$:

Property p: The mapping f is continuous when σ is taken as domain topology and τ as range topology.

Property q: The mapping f is an open mapping when σ is taken as domain topology and τ as range topology.

It is obvious that p and q are, respectively, a lower property and an upper property in $A \otimes \bar{A}$. We assert the

Theorem. *The Properties p and q just defined are saddle-point properties in $A \otimes \bar{A}$.*

Proof. The proof is given in the following two lemmas. First we observe that there is a class of saddle-points independent of f .⁵⁾ Let τ be any topology in S defined as follows: there is a cardinal number m which is infinite and a proper subset G of S is open under τ if and only if its complement G^* has a cardinal number less than m . Then (τ, τ) has properties p and q .

Lemma 1. *If (σ, τ) has properties p and q and if $(\alpha, \beta) > (\sigma, \tau)$, then (α, β) does not have property p .*

Proof. Assume the hypotheses. Now, $\alpha \geq \sigma$ and $\beta \leq \tau$ and one of these inequalities is strict. Suppose first that $\alpha > \sigma$. Let G be a set open under σ and not open under α . Then $f(G)$ is open under τ , since (σ, τ) has property q , and is open under β , since $\beta \leq \tau$. But $G = f^{-1}(f(G))$ is not open under α and f fails to be continuous in (α, β) . Suppose on the other hand that $\beta < \tau$. Let H be a set open under β and not open under τ . Since H is not open under τ and (σ, τ) has property q , $f^{-1}(H)$ is not open

⁵⁾ Advanced Problem 4548, *Amer. Math. Monthly*, **60** (1953), p. 482.

under σ , and, hence, is not open under α , since $\alpha \cong \sigma$. Thus, f fails again to be continuous in (α, β) .

From a proof dual to that of Lemma 1 we have

Lemma 2. *If (σ, τ) has properties p and q and if $(\alpha, \beta) < (\sigma, \tau)$ then (α, β) does not have property q .*

Finally we note that if G is a subset of S with $f(G) = G$ and if we take as γ_G the topology in S with G and all complements of finite sets taken as sub-base of open sets, then (γ_G, γ_G) is a saddle-point.

(Received October 26, 1953.)