On a finiteness criterion for modules.

By T. SZELE in Debrecen.

To my dear Father on his 70th birthday.

A very deep unsolved problem[1]) in group theory queries: Is a group finite, if it satisfies both the ascending and the descending chain condition[2]) for subgroups?

The answer is affirmative in case of abelian groups. In what follows we investigate the corresponding problem for modules and rings. We obtain a necessary and sufficient condition for an arbitrary ring R with unit element to possess the property that any unitary R-module[3]) is finite provided it satisfies both chain conditions for submodules:

Theorem 1. *For an arbitrary ring R with unit element the following requirements are equivalent:*

α) if a unitary R-module G satisfies both chain conditions, then G is finite;

β) every maximal left ideal of R has a finite index in R;

γ) every minimal R-module is finite;

δ) any maximal submodule of a unitary R-module G has a finite index in G.

For commutative rings R condition β) is obviously equivalent to the requirement that R must have no homomorphic image which is an infinite field. There exists a great variety of rings satisfying this condition. Important

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[1]) See [3]. — Numbers in brackets refer to the Bibliography at the end of this article.

[2]) The ascending chain condition for subgroups of a group G requires that each (strictly) ascending sequence of subgroups of G contains only a finite number of terms. The descending chain condition is defined in a similar way.

[3]) We always consider left R-modules. An R-module G is said to be unitary if the unit element of R acts as the identity operator on G. By a minimal module (called also irreducible or simple module) we mean a module $\neq 0$ which contains no proper submodule other than 0.

examples are the ring of integers in an algebraic number field and the ring of p-adic integers.

We show, moreover, that the corresponding problem for rings has an affirmative answer:

Theorem 2. *If a ring R satisfies both chain conditions for subrings, then R is finite.*

We remark that an analogous statement for rings with both chain conditions for left ideals cannot be true as the example of an infinite field shows. It can also be verified by trivial examples that neither the ascending chain condition for subrings nor the descending chain condition for subrings alone implies the finiteness of a ring.[4]) However one has the following

Theorem 3. *If a skew field S satisfies the ascending chain condition for subrings, then S is finite.*

PROOF OF THEOREM 1. *α) implies β).* Indeed, if M is a maximal left ideal in R, then the factor module R/M (regarded as a left R-module) is minimal and thus satisfies both chain conditions. Then, however, α) implies the finiteness of R/M.

β) implies γ). Let A be an arbitrary minimal R-module. We may suppose A to be a unitary module since, in the contrary case, A is a trivial module containing a prime number of elements. Now we have $A = Ra$ for an arbitrary element $a \neq 0$ of R and thus

$$A \approx R/M \tag{1}$$

where the left ideal M of R consists of all elements $x \in R$ with $xa = 0$. We infer from the minimality of A that M is a maximal left ideal in R. So (1) and β) imply the finiteness of A.

γ) implies δ), since the factor module G/B is minimal provided B is a maximal submodule of G.

δ) implies α). Let G be an infinite unitary R-module possessing property δ). We show that then G cannot satisfy both chain conditions simultaneously. If there exists a submodule $D \neq 0$ in G which contains no (in D) maximal submodule, then it is obvious that D — and a fortiori G — does not satisfy the ascending chain condition. Thus we can suppose that any submodule $\neq 0$ of G contains a maximal submodule. Then there exists a strictly descending chain

$$G = G_0 \supset G_1 \supset G_2 \subset \cdots \tag{2}$$

of submodules in G such that G_n is maximal in G_{n-1} $(n = 1, 2, 3, \ldots)$. Hence,

[4]) Such examples yield the ring of rational integers resp. the zero-ring with PRÜFER's quasicyclic group of type (p^∞) as additive group, the latter being defined as the additive group modulo 1 of all rational numbers with p-power denominators for a fixed prime p. (A ring R is called a zero-ring if the product of any two elements in R is 0.)

by δ), G_n has a finite index in G_{n-1} and so the chain (2) is infinite. Thus we have obtained that G does not satisfy the descending chain condition which completes the proof of Theorem 1.

PROOF OF THEOREM 3. Let S be a skew field satisfying the ascending chain condition for subrings. Then S is of characteristic $p>0$, since the rational number field does not satisfy the ascending chain condition for subrings. Now we show that the skew field S is absolutely algebraic, i. e., every element of S is algebraic over the prime field P_p (with p elements) of S. This will complete the proof since then, by an important theorem of JACOBSON[5]), S is commutative, i. e., S is a subfield of the algebraic closure of the prime field P_p and, as such, it satisfies the ascending chain condition for subrings (or even for subfields) only in the case S is finite.

In order to prove that the skew field S is absolutely algebraic, we suppose that S contains an element x transcendental over the prime field P_p. Then we get that the simple extension field $P_p(x)$ which is isomorphic to the field of rational functions in x over the base field P_p, does not satisfy the ascending chain condition for subrings. As a matter of fact, let $g_1(x), g_2(x), \ldots$ be an infinite set of distinct irreducible polynomials in x over P_p with leading coefficient 1, and let R_k be the subring of $P_p(x)$ consisting of all rational functions with denominators of the form

$$g_1(x)^{m_1}\cdot g_2(x)^{m_2}\cdot\ldots\cdot g_k(x)^{m_k} \qquad (m_i \geqq 0;\ \ i=1,2,\ldots,k).$$

Thus $R_1 \subset R_2 \subset \ldots$ is an infinite ascending chain of subrings in S, in contradiction to our assumption. This completes the proof of Theorem 3.

PROOF OF THEOREM 2. Let R be an arbitrary ring satisfying both chain conditions for subrings. Then, in particular, R satisfies the descending chain condition for left ideals, i. e., R is an Artinian ring. By a well-known theorem the radical U of R — which can be defined in this case as the union of all nilpotent left ideals of R — is a nilpotent twosided ideal in R of exponent, say n, i. e., n is the least natural number such that $U^n=0$. Since the factor ring R/U contains no non-zero nilpotent left ideal and satisfies the descending chain condition for left ideals, R/U is isomorphic, by the classical WEDDERBURN-ARTIN structure theorem, to a direct sum of a finite number r of complete matrix rings over skew fields S_i $(i=1,2,\ldots,r)$. Therefore in order to prove the finiteness of R/U it is sufficient to show that each of these basic skew fields S_i is finite. But the ring R/U contains a subring isomorphic to S_i. On the other hand, the ring R/U, as a homomorphic image of R, satisfies both chain conditions for subrings, and thus also S_i satisfies these conditions. So the finiteness of S_i follows from Theorem 3.

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[5]) See Theorem 2 on p. 701 in [2]. For a very simple and throughly elementary proof of this theorem see [1].

We complete the proof of Theorem 2 by showing that the radical U of R is a finite ring. Since U satisfies both chain conditions for subrings, in the case $n=2$ (i. e., if U is a zero-ring) this follows from the fact that an abelian group is finite if it satisfies both chain conditions for subgroups. In the sequel we use induction with respect to the exponent n of U. Let U be a nilpotent ring with exponent n satisfying both chain conditions for subrings, and suppose that the finiteness of such a ring with an exponent less than n is already proved. Then the ring U^{n-1} as well as the factor ring U/U^{n-1} are finite rings since both are nilpotent rings with exponents less than n and satisfying the chain conditions. But this implies the finiteness of U. Thus also Theorem 2 is proved.

ADDED IN PROOF (March 8, 1955): Professor A. G. KUROŠ has kindly directed my attention to the fact that Theorem 2 is already proved in a paper of V. I. SNEJDMYULLER [*Mat. Sbornik N. S.* **27** (69) (1950), 219—228].

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