

On the complete direct sum of countable abelian groups.

By JERZY ŁOŚ in Toruń.

A group G is called decomposable into a discrete direct sum of groups H_p , $p \in P$, if it is isomorphic to the discrete direct sum $\sum_{p \in P} H_p$.

The purpose of this paper is to show that a complete direct sum $\sum_{t \in T}^* G_t$ of countable abelian torsion free groups G_t is decomposable into a discrete direct sum of countable groups H_p if and only if almost all groups G_t , with exception of a finite number of them, are divisible groups. This theorem is a generalization of the known result of A. P. MISHINA [2].

1. Terms and notations. All groups are additively written abelian groups.

$\{G_t\}_{t \in T}$ being a family of groups, we denote by $S^* = \sum_{t \in T}^* G_t$ the complete direct sum (c.d.s.) of the groups G_t , i.e. the group of all functions x defined on T and such that $x(t) = x_t \in G_t$ for every t in T . By $S = \sum_{t \in T} G_t$ we denote the subgroup of S^* which consists of all these functions $x \in S^*$ for which the set $\Theta(x) = \{x_t \neq 0\}_{t \in T}$ is finite. S is called the discrete direct sum (d.d.s.) of the groups G_t . If T is a finite set, we have $S^* = S$; in this case S is called simply the direct sum of G_t . For the direct sum of two groups G_1 and G_2 we write $G_1 + G_2$. If $G = G_1 + G_2$ then G_1 and G_2 are called direct summands in G .

Let x be in S^* and X a subset of T , then by $x|X$ we denote the (uniquely determined) element y in S^* , for which $y_t = x_t$ with t in X and $y_t = 0$ with t in $T - X$.

Let G be an abelian group and g an element in G . The element g is called divisible in G by an integer n , if there exists an x in G with $nx = g$. If every g in G is divisible in G by every integer n , then G is called a divisible group. If G has no (non-zero) divisible subgroups, then G is called a reduced group.

If X and Y are subsets of the set T , then by the symmetric difference of X and Y we understand the set $X \dot{-} Y = X \cap (T - Y) + (T - X) \cap Y$. If K is a field of sets and I an ideal in K , then by K/I we denote the family of partition classes of K corresponding to the relation $X - Y \in I$.

2. Lemmas. The following lemmas are well known:

(2.1) (BAER [1]) Every abelian group G is decomposable into a direct sum $G = G_1 + G_2$, where G_1 is a divisible group and G_2 a reduced group.

(2.2) (SZELE [3]) Every divisible abelian group is decomposable into a d.d.s. of countable groups.

(2.3) Every d.d.s. and every c.d.s. of divisible groups is a divisible group.

(2.4) Every d.d.s. and every c.d.s. of reduced groups is a reduced group.

(2.5) A subgroup of a reduced group is a reduced group.

(2.6) If $G = G_1 + G_2$, G_2 is a reduced group, and G_0 is a divisible subgroup of G , then G_0 is a subgroup of G_1 .

(2.7) If $\{G_\xi\}_{\xi \in \Xi}$ is a family of divisible subgroups of a group G , then the group G_0 generated by the union of all G_ξ 's is a divisible group.

(2.8) If $G = G_1 + G_2$ and G_0 is a subgroup of G_1 , then $G/G_0 = G_1/G_0 + G_2$.

(2.9) If $S = \sum_{t \in T} G_t$ and S_0 is a countable subgroup of S , then there exists a countable subset T_0 of T such that S_0 is a subgroup of $\sum_{t \in T_0} G_t$.

(2.10) Let $S^* = \sum_{t \in T}^* G_t$ and $S = \sum_{t \in T} G_t$. For x and y in S^* , if $\Theta(x) \dot{-} \Theta(y)$ is infinite, then $x + S \neq y + S$.

3. A theorem on S^*/S .

Theorem 1. If G_n are (non-zero) abelian torsion free groups and $S^* = \sum_{n=1}^{\infty} G_n$, $S = \sum_{n=1}^{\infty} G_n$, then there exists in S^*/S a divisible subgroup of the power of continuum ($2^{\aleph_0}$).

PROOF. For every n , let g_n denote a non-zero element in G_n and for integers k, l, n with $l \leq n$, let $g(n, k, l)$ denote an element y in S^* such that $y_m = 0$ for $m < n$, and $y_m = \frac{m!k}{l} g_m$ for $m \geq n$. Evidently the elements $g(n, k, l) + S$ form a divisible subgroup of S^*/S . We shall denote this subgroup by R .

Let now K be the field of all subsets of the set of natural numbers and I the ideal of finite sets in K . We have $\bar{K} = 2^{\aleph_0}$, $\bar{I} = \aleph_0$, therefore $\bar{K}/\bar{I} = 2^{\aleph_0}$. By the axiom of choice, there exists a family of sets of natural numbers $\{X_\xi\}_{\xi \in \Xi}$ of the power of continuum and such that $\overline{X_{\xi_1} \dot{-} X_{\xi_2}} = \aleph_0$, for $\xi_1 \neq \xi_2$.

We set now $x^{(\xi)} = g(1, 1, 1) + X_\xi$. We have evidently $\Theta(x^{(\xi)}) = X_\xi$. Therefore $\overline{\Theta(x^{(\xi_1)}) \dot{-} \Theta(x^{(\xi_2)})} = \aleph_0$ and by (2.10) all $x^{(\xi)} + S$ are different for different ξ in Ξ . For every ξ in Ξ let R_ξ be the group of all elements $y + X_\xi$ with

y in R . R_ξ as homomorphic image of R is a divisible group and evidently contains $x^{(\xi)} + S$. By (2.7) there exists a divisible subgroup of S^*/S which contains all the elements $x^{(\xi)} + S$ and therefore is of the power of continuum.

4. The main theorems.

Theorem 2. *If T is an infinite set and G_t for t in T are reduced abelian torsion free countable groups then the c.d.s. $S^* = \sum_{t \in T}^* G_t$ is not decomposable into a d.d.s. of countable groups.*

PROOF. Let T_0 be a countable subset of T , $S_0^* = \sum_{t \in T_0}^* G_t$, $S_0 = \sum_{t \in T_0} G_t$, $S_1^* = \sum_{t \in T - T_0}^* G_t$. We have $S^* = S_0^* + S_1^*$ and S_0 is a subgroup of S_0^* . Therefore by (2.8) $S^*/S_0 = S_0^*/S_0 + S_1^*$ and by Theorem 1, there exists in S^*/S_0 a divisible subgroup R of the power of continuum. Suppose now S^* is decomposable into the d.d.s. of countable groups H_p : $S^* = \sum_{p \in P} H_p$. Then by $\bar{S}_0 = \aleph_0$ and (2.9) there exists a countable subset P_0 of P , such that S_0 is a subgroup of $\sum_{p \in P_0} H_p$. Therefore $S^*/S_0 = (\sum_{p \in P_0} H_p)/S_0 + \sum_{p \in P - P_0} H_p$. In view of (2.4) and (2.5) $\sum_{p \in P - P_0} H_p$ is a reduced group and in view of (2.6) R is a subgroup of $(\sum_{p \in P_0} H_p)/S$, contrary to the fact that $\sum_{p \in P_0} H_p$ is a countable group and R is of the power of continuum.

Theorem 3. *A c.d.s. of countable abelian torsion free groups is decomposable into a d.d.s. of countable groups if and only if almost all summands, with exception of a finite number of them, are divisible groups.*

This follows from (2.3), (2.2), (2.1) and Theorem 2.

5. Remarks. It is evident, that the proof of Theorem 1 may be reconstructed also for arbitrary abelian groups under the assumption that

(5.1) *for every n there exists in G_n an element g_n of order greater than $n!$.*

If in the family of groups $\{G_t\}_{t \in T}$ there may be found a sequence of different groups $G_{t_1}, G_{t_2}, \dots$, such that (5.1) is fulfilled, then $\{G_t\}_{t \in T}$ is called a family of essentially unbounded order.

We have the following theorems:

Theorem 4. *If $\{G_t\}_{t \in T}$ is a family of abelian groups of essentially unbounded order, $S^* = \sum_{t \in T}^* G_t$ and $S = \sum_{t \in T} G_t$, then in S^*/S there is contained a divisible subgroup of the power of continuum.*

Theorem 5. *If $\{G_t\}_{t \in T}$ is a family of reduced countable (or finite) abelian groups with essentially unbounded order, then the c.d.s. $S^* = \sum_{t \in T}^* G_t$ is not decomposable into a d.d.s. of countable (or finite) groups.*

Bibliography.

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