

Capturability of a pseudo differential game of pursuit

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Abstract. This paper is concerned with the capturability of a pseudo linear differential game of pursuit. The pursuit set that the pursuit will end once the initial state lies in this set is given by the method of integration of a multi-valued function. The result obtained here solves a Pontrjagin's problem on the linear differential game of pursuit, but the requirements of the convexity of the control set and Pontrjagin's other related conditions are removed.

1. Problems and Results

In PONTRJAGIN [1–4], the following linear pursuit differential game was investigated:

$$(1) \quad \begin{cases} \frac{dz(t)}{dt} = Cz(t) - u(t) + v(t), & z \in \mathbb{R}^n, u \in P, v \in Q, \\ z(0) = z_0, \end{cases}$$

where the control sets P and Q are two compact convex sets of the Euclidean space $\mathbb{R}^n$, and the terminal set is assumed to be a linear subspace of $\mathbb{R}^n$. In [1], a approximate capturability is given. In [3–4], a complicated method is used to obtain the capturability under strong assumptions. In this paper, we study a general pseudo linear differential game of pursuit described by

$$(2) \quad \begin{cases} \frac{dz(t)}{dt} = Cz(t) + f(u(t), v(t)), & z \in \mathbb{R}^n, u \in P \subset \mathbb{R}^p, \\ & v \in Q \subset \mathbb{R}^q, \\ z(0) = z_0, \end{cases}$$

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where C is a $n \times n$ matrix and the control sets P and Q are assumed to be compact in the corresponding Euclidean spaces; the vector function f is continuous in $P \times Q$; $u(t)$ and $v(t)$ are Lebesgue measurable functions of t . The terminal set M is assumed to be a linear subspace of $\mathbb{R}^n$.

First, we state some definitions and known preliminary results.

Definition 1 [1,2]. For a given initial state z_0 , the differential game (2) is said to be capturable within time τ if for any measurable function $v(t) \in Q$, $t \geq 0$, there exist a time $\bar{t} \leq \tau$ and a measurable function $u(t)$ the value of which at time t is determined by $\{z(t), v(s), 0 \leq s \leq t\}$ such that $x(\bar{t}) \in M$, where $z(t)$ is the solution of Eq. (2) corresponding u and v .

Definition 2 [6]. Let $F(t)$ be a multi-valued function defined in $[\alpha, \beta]$, $F(t) \in k(\mathbb{R}^n)$, the set of all non-empty compact subsets of $\mathbb{R}^n$. The integration of $F(t)$ in $[\alpha, \beta]$, denoted by $\int_{\alpha}^{\beta} F(t)dt$, is defined as following:

$$\int_{\alpha}^{\beta} F(t)dt = \left\{ z : z = \int_{\alpha}^{\beta} f(t)dt, f(t) \in F(t), f(t) \text{ is integrable in } [\alpha, \beta] \right\}.$$

Theorem 1 [6]. Let $\int_{\alpha}^{\beta} F(t)dt$ be the integration of the multi-valued function $F(t)$ in $[\alpha, \beta]$. Then

- (i). $\int_{\alpha}^{\beta} F(t)dt$ is compact in $\mathbb{R}^n$;
- (ii). in the sense of Hausdorff metric, if

$$\lim_{i \rightarrow \infty} F_i(t) = F(t), \quad \text{for any } t \in [\alpha, \beta],$$

then

$$\lim_{i \rightarrow \infty} \int_{\alpha}^{\beta} F_i(t)dt = \int_{\alpha}^{\beta} F(t)dt.$$

Theorem 2 (Filippov Lemma) [7]. Let $f(t, u) \in \mathbb{R}^n$ be continuous with respect to $(t, u) \in [\alpha, \beta] \times \mathbb{R}^r$. Let $Q(t)$ be a continuous multi-valued function defined in $[\alpha, \beta]$ with value of bounded closed set and $y(t)$ be such a measurable $n \times 1$ vector-valued function that

$$y(t) \in f(t, Q(t)), \quad \text{for } t \in [\alpha, \beta] \text{ a.e.}$$

Then there exists a $r \times 1$ measurable vector-valued function $u(t) \in Q(t)$ such that

$$f(t, u(t)) = y(t) \quad \text{for } t \in [\alpha, \beta] \text{ a.e.}$$

Our main result is the following

Theorem 3. *If for any time t*

$$S(t) = \bigcap_{v \in Q} \Pi e^{tC} f(P, v) \neq \emptyset,$$

then the differential game (2) is capturable within time τ for any initial state z_0 satisfying

$$\Pi e^{tC} z_0 \in - \int_0^\tau S(t) dt,$$

where Π is the orthogonal projection of $\mathbb{R}^n$ to the orthogonal complement L of M , e^{tC} is the semigroup generated by matrix C and $\int_0^\tau S(t) dt$ denotes the integration of the multi-valued function $S(t)$ in $[0, \tau]$.

For the linear case of system (2), $f(u, v) = -u + v$ and

$$S(t) = \bigcap_{v \in Q} \Pi e^{tC} f(P, v) = - [\Pi e^{tC} P \ast \Pi e^{tC} Q],$$

where $A \ast B$ denotes the geometric difference of the sets A and B . From Theorem 3, we have immediately the following

Corollary. *If for any time t*

$$\Pi e^{tC} P \ast \Pi e^{tC} Q \neq \emptyset,$$

then the differential game (1) is capturable within time τ for any initial state z_0 satisfying

$$\Pi e^{tC} z_0 \in \int_0^\tau [\Pi e^{tC} P \ast \Pi e^{tC} Q] dt.$$

Remark. To ensure the same conclusion as our Corollary, it was further assumed in [3,4] that $\Pi e^{tC} P \ast \Pi e^{tC} Q$ and L have the same dimension and

- (a) $\int_0^\tau S(t) dt$ is convex;
- (b) $\int_0^\tau S(t) dt$ has smooth boundary;
- (c) there is no any line segment on the boundary of $\int_0^\tau S(t) dt$.

It is seen that all these conditions are removed here.

2. The proof of the Theorem 3

In what follows, the continuity of the multi-valued function is with respect to the Hausdorff metric.

First notice that for any given $v \in Q$, $f(u, v)$ is continuous on the compact set P , and hence $\Pi e^{tC} f(P, v)$ and $\bigcap_{v \in Q} \Pi e^{tC} f(P, v)$ are compact.

Since $\Pi e^{tC} z_0 \in -\int_0^t S(s) ds$, by Definition 2, one can find a measurable function $x(t) \in S(t)$, $t \in [0, \tau]$ such that

$$\Pi e^{tC} z_0 = -\int_0^t x(s) ds.$$

Let $v(t) \in Q \subset \mathbb{R}^q$ be a measurable function defined in $[0, \tau]$. Then the fact that

$$x(\tau - t) \in S(\tau - t) = \bigcap_{v \in Q} \Pi e^{(\tau-t)C} f(P, v(t))$$

leads directly to $x(\tau - t) \in \Pi e^{(\tau-t)C} f(P, v)$. Since $\Pi e^{(\tau-t)C} f(u, v(t))$ is measurable for $t \in [0, \tau]$ and continuous for $u \in P$, by Filippov's Lemma (Theorem 2), there exists a measurable function $u(t) \in P$, $t \in [0, \tau]$ such that

$$x(\tau - t) = \Pi e^{(\tau-t)C} f(u(t), v(t)), \quad \text{for } t \in [0, \tau] \text{ a.e.}$$

and the value of $u(t)$ at time t is determined only by $v(t)$ at the same time moment.

By these $u(t)$ and $v(t)$, the value of the solution $z(t)$ of Eq. (2) at time τ will be

$$z(\tau) = e^{\tau C} z_0 + \int_0^\tau e^{(\tau-t)C} f(u(t), v(t)) dt,$$

and hence

$$\begin{aligned} \Pi z(\tau) &= \Pi e^{\tau C} z_0 + \int_0^\tau \Pi e^{(\tau-t)C} f(u(t), v(t)) dt, \\ &= \Pi e^{\tau C} z_0 + \int_0^\tau x(\tau - t) dt \\ &= \Pi e^{\tau C} z_0 + \int_0^\tau x(t) dt = 0. \end{aligned}$$

The proof is thus complete.

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References

- [1] L. S. PONTRJAGIN, Linear differential games I, *Dokl. Akad. Nauk. USSR* **174** (1967), 1278–1280.
- [2] L. S. PONTRJAGIN, Linear differential games of pursuit, *Mat. Sb.* **112(154)** (1980), 307–330; English Transl. in *Math. USSR Sb.* **40** (1981).
- [3] L. S. PONTRJAGIN and A. S. MISHENKO, Solution of the linear differential games of pursuit without discriminable escape object, *Dokl. Akad. Nauk. USSR* **277 No. 5** (1984), 1063–1066.
- [4] L. S. PONTRJAGIN and A. S. MISHENKO, Linear differential games of pursuit (analytical theorem), *Mat. Sb.* **131(173)** (1986), 131–158.
- [5] L. S. PONTRJAGIN and L. EBANNOV, The measurable strategy in differential games, *Mat. Sb.* **180** (1989), 119–135.
- [6] V. I. VLAGODATSIH and A. F. FILIPPOV, Differentiable inclusion and optimal control, *Memoirs of Mathematical Institute, Academy of Science, SSSR* **169** (1985), 194–252.
- [7] R. GABASOV and F. KIRILLOVA, *The Qualitative Theory of Optimal Processes*, Marcel Dekker, INC. NY., 1976, pp. 181–182.

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