Symmetric matrices, quadratic forms and linear constraints.

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The condition for a quadratic form $\mathbf{x}'A\mathbf{x}$ to be positive definite subject to restrictive linear conditions $U'\mathbf{x}=\mathbf{0}$[1]) is fundamental for the investigation of maxima and minima of functions which are restricted by relations between their variables and for a variety of other questions. However, the existing proofs for establishing these criteria do not achieve all the directness which is possible, and which is desirable for such a result. The object here is to give a new derivation; and, following DEBREU, it is founded on a consideration of the pencil $A+\lambda UU'$.

Theorem 1. *If A is a real symmetric matrix of order n, and δ_r its leading principal minor of order r, and if $\delta_r \neq 0$ $(r=1,\dots,n)$, then there exists an upper triangular matrix T, with unit elements on the diagonal, such that $T'AT=D$, where D is the diagonal matrix with r th element $d_r=\delta_r/\delta_{r-1}$.*[2])

Take as hypothesis the validity of this theorem for matrices of order n. We shall deduce it for matrices of order $n+1$. Then, since it is true in the case of order one, its proof by induction will be complete. Thus, let $\begin{pmatrix} A & \mathbf{a} \\ \mathbf{a}' & a \end{pmatrix}$ be the considered matrix of order $n+1$, where A is of order n. By hypothesis we can find the triangular matrix T with $T'AT=D$ as required. Then, since $\begin{vmatrix} A & \mathbf{a} \\ \mathbf{a}' & a \end{vmatrix} / |A| = a-\mathbf{a}'A^{-1}\mathbf{a}$, by the Cauchy determinant expansion, the extension to order $n+1$ is directly shown by the relation

$$\begin{pmatrix} T' & \mathbf{0} \\ (-A^{-1}\mathbf{a})' & 1 \end{pmatrix} \begin{pmatrix} A & \mathbf{a} \\ \mathbf{a}' & a \end{pmatrix} \begin{pmatrix} T & -A^{-1}\mathbf{a} \\ \mathbf{0}' & 1 \end{pmatrix} = \begin{pmatrix} D & \mathbf{0} \\ \mathbf{0}' & a-\mathbf{a}'A^{-1}\mathbf{a} \end{pmatrix}.$$

Theorem 2. *The quadratic form $\mathbf{x}'A\mathbf{x}$ is positive definite if and only if $\delta_r>0$ $(r=1,\dots,n)$.*

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1) This condition was first stated and proved by H. B. MANN [1]. The author has given another proof in [2]; and a further one is suggested by G. DEBREU in [3]. Here A, U denote matrices of order $n\times n$, $n\times m$ and $\mathbf{x}$ denotes a vector of order n.

2) cf. TURING [6].

If $\mathbf{x}'A\mathbf{x}$ is positive definite, then $\delta_r \neq 0$ $(r=1,\ldots,n)$; otherwise there would exist a vector $\mathbf{x} \neq 0$ such that $A\mathbf{x}=\mathbf{0}$, and $\mathbf{x}'A\mathbf{x}=0$. Hence, by Theorem 1, there exists a regular matrix T such that $T'AT=D$, which is positive definite with A, is diagonal, with elements $d_r=\delta_r/\delta_{r-1}$, which are all positive; and thus $\delta_r>0$ $(r=1,\ldots,n)$. Conversely, if $\delta_r>0$ $(r=1,\ldots,n)$, then there exists, by Theorem 1, a regular matrix T such that $T'AT=D$ is the diagonal matrix with elements $d_r=\delta_r/\delta_{r-1}$. But all these elements are positive, so that D and hence A is positive definite.[3])

Theorem 3. *If $\mathbf{x}'A\mathbf{x}$ is definite subject to $U\mathbf{x}=\mathbf{0}$, and if λ_0 is the least root of the equation $|A-\lambda UU'|=0$, then $\mathbf{x}'(A-\lambda^* UU')\mathbf{x}$ is positive definite for all $\lambda^*<\lambda_0$.*

The stationary values of $\lambda_{\mathbf{x}}=\mathbf{x}'A\mathbf{x}/\mathbf{x}'UU'\mathbf{x}$ are attained where $(A-\lambda_{\mathbf{x}}UU')\mathbf{x}=\mathbf{0}$, as appears by methods of the calculus. It follows that the minimum value λ_0 of $\lambda_{\mathbf{x}}$ is the least root of $|A-\lambda UU'|=0$. Now for any $\lambda^*<\lambda$ we have

$$\mathbf{x}'(A-\lambda^* UU')\mathbf{x}=\mathbf{x}'(A-\lambda_0 UU')\mathbf{x}+(\lambda_0-\lambda^*)\mathbf{x}'UU'\mathbf{x} \geqq 0,$$

where $\mathbf{x}'(A-\lambda_0 UU')\mathbf{x}$ and $(\lambda_0-\lambda^*)\mathbf{x}'UU'\mathbf{x}$ are non-negative definite. Since any non-negative quantities whose sum is zero are each zero, the equality here is attained only where $\mathbf{x}'(A-\lambda_0 UU')\mathbf{x}=0$ and $(\lambda_0-\lambda^*)\mathbf{x}'UU'\mathbf{x}=0$, or equivalently where $\mathbf{x}'A\mathbf{x}=0$ and $U'\mathbf{x}=\mathbf{0}$. Thus it appears that if $\mathbf{x}'A\mathbf{x}\neq 0$ whenever $U'\mathbf{x}=\mathbf{0}$ and $\mathbf{x}\neq\mathbf{0}$, which is the case when $\mathbf{x}'A\mathbf{x}$ subject to $U'\mathbf{x}=\mathbf{0}$ is definite, either positive or negative, then $\mathbf{x}'(A-\lambda^* UU')\mathbf{x}$ is positive definite.

Theorem 4. *A necessary and sufficient condition that $\mathbf{x}'A\mathbf{x}$ be positive definite subject to $U'\mathbf{x}=\mathbf{0}$ is that $\mathbf{x}'(A+\lambda UU')\mathbf{x}$ be positive definite for large λ.*

The necessity follows from Theorem 3, by which $\mathbf{x}'(A+\lambda UU')\mathbf{x}$ is positive definite for $\lambda>|\lambda_0|$, and the sufficiency is obvious.

The linear constraints given by $U'\mathbf{x}=\mathbf{0}$ are supposed independent so that U has all its m columns independent. Accordingly U also has some m of its rows independent, and without loss in generality these can be supposed leading; so if U_r denotes the submatrix of U formed from its leading r rows then $|U_m|\neq 0$.

Let A_r denote the leading principal square submatrix of A of order r.

Theorem 5. *A necessary and sufficient condition that $\mathbf{x}'A\mathbf{x}$ be positive definite subject to $U'\mathbf{x}=\mathbf{0}$ is that*

$$\triangle_r=(-1)^r\begin{vmatrix} 0 & U_r' \\ U_r & A_r \end{vmatrix}>0 \qquad (r=m+1,\ldots,n),$$

provided $|U_m|\neq 0$.

[3]) cf. Goddard [5].

Call $\mathbf{x}'A\mathbf{x}$ positive definite subject to $U'\mathbf{x}=\mathbf{0}$ the condition P, and $\triangle_r>0$ $(r=m+1,\ldots,n)$ the condition $\triangle$. By Theorem 4, P is equivalent to $\mathbf{x}'(A+\lambda UU')\mathbf{x}$ positive definite for large λ, and by Theorem 2 this is equivalent to $|A_r+\lambda U_r U_r'|>0$ $(r=1,\ldots,n)$ for large λ. But, again by Theorem 2, $|A_r+\lambda U_r U_r'|>0$ $(r=1,\ldots,m)$ is equivalent to $\mathbf{x}_m'(A_m+\lambda U_m U_m')\mathbf{x}_m$ positive definite, which is in any case true for large λ, since $\mathbf{x}_m' U_m U_m' \mathbf{x}_m$ is positive definite, with $|U_m|\neq 0$. Thus P is equivalent to $|A_r+\lambda U_r U_r'|>0$ $(r=m+1,\ldots,n)$ for large λ. But, for $r=m+1,\ldots,n$,

$$\begin{aligned}|A_r+\lambda U_r U_r'| &= \sum_{s=0}^{r}\lambda^s \text{ trace } A_r^{[s]}(U_r U_r')^{(s)} = \\ &= \sum_{s=0}^{r}\lambda^s \text{ trace } U_r^{(s)\prime} A_r^{[s]} U_r^{(s)} = \\ &= o(\lambda^r)+\lambda^r(-1)^r \begin{vmatrix} 0 & U_r' \\ U_r & A_r \end{vmatrix} \qquad (\lambda\to\infty),\end{aligned}$$

by the Binet—Cauchy theorem, and by a double Laplace expansion of the bordered determinant relative to the bordering rows and columns.[4]) Thus on the hypothesis $\triangle_r\neq 0$ $(r=m+1,\ldots,n)$ we have P equivalent to $\triangle$. But this hypothesis is explicitly contained in $\triangle$. Moreover it is implied by P. For should we have $\triangle_r=0$ for some $r=m+1,\ldots,n$, the equations

$$U_r'\mathbf{x}_r=\mathbf{0}$$
$$U_r\mathbf{y}_m+A_r\mathbf{x}_r=\mathbf{0}$$

would have a non-null solution where $\mathbf{x}_r$, $\mathbf{y}_m$ denote vectors of order r, m respectively; and this would be such that $\mathbf{x}_r\neq\mathbf{0}$ since $U_r\mathbf{y}_m=\mathbf{0}$ implies $\mathbf{y}_m=\mathbf{0}$, for $r>m$; and also such that $\mathbf{x}_r'A\mathbf{x}_r=0$ and $U_r'\mathbf{x}_r=\mathbf{0}$, in contradiction to P. Thus P and $\triangle$ are shown to be equivalent, on the hypothesis that either hold; and thus they are equivalent.

It appears thus that the quadratic forms $\mathbf{x}'A\mathbf{x}$ which are positive definite subject to $U'\mathbf{x}=\mathbf{0}$ form the open region defined by the open inequalities $\triangle_r>0$ $(r>m)$, whose closure, which consists of the quadratic forms $\mathbf{x}'A\mathbf{x}$ wich are non-negative definite subject to $U'\mathbf{x}=\mathbf{0}$, is accordingly defined by the closed inequalities $\triangle_r\geqq 0$ $(r>m)$.

Since $\mathbf{x}'(-A)\mathbf{x}$ is positive when $\mathbf{x}'A\mathbf{x}$ is negative, the condition for $\mathbf{x}'A\mathbf{x}$ to be negative definite subject to $U'\mathbf{x}=0$ is obtained when A is replaced by $-A$ in Theorem 5, that is to say as $(-1)^r\triangle_r>0$ $(r>m)$.

[4]) See Aitken [4], pp. 102, 83 and 93 respectively in connection with the successive steps here.

Bibliography.

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