On pure subgroups and direct summands of abelian groups.

To Professor László Kalmár on his 50th birthday.

By S. GACSÁLYI in Debrecen.

Le G be an additive abelian group. A subgroup A of G is said to be a pure subgroup[1]) in G if any equation $nx = a \in A$ which can be solved in G, possesses also a solution $x \in A$. (Here n denotes an arbitrary natural number.) Obviously every direct summand of G is a pure subgroup in G. The converse statement, however, is not true in general. For instance, the torsion subgroup (i.e. the subgroup consisting of all elements of finite order) of G is always a pure subgroup in G, without being necessarily a direct summand of G, as it is well-known. Thus the concept of direct summand is more restrictive than that of pure subgroup. In what follows we give characterizations of similar nature for these concepts, namely both in terms of solvability of equation systems. These characterizations exhibit also, how much "stronger" the requirement to be a direct summand is for a group than that of being a pure subgroup.

We make use of the following important theorem of L. KULIKOV [3]: If A is a pure subgroup in an abelian group G such that G/A is a direct sum of cyclic groups, then A is a direct summand of G. The proof of this theorem can be obtained by the following criterion: A is a pure subgroup of G if and only if in the factor group G/A the order of each coset of G relative to A coincides with the order of some element of G belonging to this coset.

The "most general" algebraic equation in the unknowns $x_1, \ldots, x_k$ over the subgroup A of G, constructed by the aid of addition as the only operation defined in G, can always be written, by the commutativity of G, in the form

$$m_1 x_1 + \cdots + m_k x_k = a \in A \tag{1}$$

[1]) In investigations on infinite abelian groups the concept of "Servanzuntergruppe", introduced and thus named by H. PRÜFER [4], has proved very important. We make use of the above denomination, which prevails in the most recent literature on the subject [1], [2]. — Numbers in brackets refer to the Bibliography at the end of this article.

where $m_1, \ldots, m_k$ are rational integers. By a system of equations over A we shall mean a set of equations of the form (1) where the cardinal number of the equations as well as that of the unknowns x_ν is arbitrary in the whole, but each single equation contains a finite number of unknowns only. We say that such a system is solvable in a subgroup B of G if one can find elements $x_\nu = b_\nu \in B$ which satisfy all equations of the system. Now we prove the following theorems:

Theorem 1. *A subgroup A of an abelian group G is a pure subgroup in G if and only if every system of equations over A in a finite set of unknowns*[2]) *$x_1, \ldots, x_k$ solvable in G possesses also a solution in A.*

REMARKS. This theorem answers a question raised by L. Fuchs. The finiteness of the whole set of unknowns occuring in the system of equations is an essential restriction in Theorem 1, and cannot be omitted. This follows from Theorem 2 and from the fact that there exist pure subgroups which are no direct summands.

Theorem 2. *A subgroup A of an abelian group G is a direct summand of G if and only if every system of equations over A solvable in G possesses also a solution in A.*[3])

REMARK. From Theorems 1 and 2 the possibility results of defining "subgroups of intermediate character" between pure subgroups and direct summands. In fact, one can define $\mathfrak{c}$-pure subgroups A of G for any given infinite cardinal $\mathfrak{c}$ by the property: every system of equations over A in a set of unknowns with cardinality less than $\mathfrak{c}$, which is solvable in G, possesses also a solution in A. So, by Theorem 1, the $\aleph_0$-pure subgroups of G coincide with the (usual) pure subgroups, and, by the following proof of Theorem 2, the $\mathfrak{c}$-pure subgroups of G for a cardinal $\mathfrak{c}$ greater than the cardinality of G coincide with the direct summands of G.

PROOF OF THEOREM 1. By the definition of pure subgroups it is sufficient to prove the "only if" statement of Theorem 1. Consider a system of equations over the pure subgroup A of G in k unknowns, solvable in G. Let $x_1, \ldots, x_k$ be elements of G giving a solution of this system. Then A is a pure subgroup also of the group $H = \{A, x_1, \ldots, x_k\}$ generated by A and the elements $x_1, \ldots, x_k$. Since, on the other hand, the factor group H/A is (finitely generated and therefore) a direct sum of cyclic groups, it follows

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[2]) The cardinality of the equations in the system is arbitrary.

[3]) By the proof of this theorem, for A to be a direct summand of G it is sufficient that the above requirement be satisfied for arbitrary systems of equations in a set of unknowns of a cardinality not greater than that of the group G.

from the theorem of KULIKOV cited above that A is a direct summand of H:

$$H = A + B.$$

Consequently, every element of H may be represented uniquely in form $a+b$ with $a \in A$, $b \in B$. Applying this in particular to the elements x_i we have

$$x_i = a_i + b_i \quad (a_i \in A,\ b_i \in B;\ i = 1, \ldots k).$$

But then the elements $a_1, \ldots, a_n$ of A form a solution of our equation system.

PROOF OF THEOREM 2. If A is a direct summand of G, then every system of equations over A solvable in G admits also a solution in A as we have just seen. Now let us suppose, conversely, that any system of equations over a given subgroup A of G solvable in G admits also a solution in A. Let $x_1, x_2, \ldots$ be a (not necessarily countable!) set of elements in G, which together with A generate the whole group G:

$$G = \{A, x_1, x_2, \ldots\}. \tag{2}$$

We consider the set of all relations of the form

$$m_1 x_{\alpha_1} + \cdots + m_k x_{\alpha_k} = a \in A. \tag{3}$$

All these relations form a system of equations over A, a solution of which is given by the set of elements $x_1, x_2, \ldots$ of G. Consequently this equation system admits also a solution $a_1, a_2, \ldots$ in A. Now we can prove that A is a direct summand[4]) of G, namely

$$G = A + B \tag{4}$$

holds for the subgroup

$$B = \{x_1 - a_1, x_2 - a_2, \ldots\}$$

generated by all elements $x_1 - a_1, x_2 - a_2, \ldots$. As a matter of fact, the subgroups A and B generate the whole group G, since $\{A, B\}$ contains $x_1, x_2, \ldots$ (see (2)). On the other hand, let

$$m_1(x_{\alpha_1} - a_{\alpha_1}) + \cdots + m_k(x_{\alpha_k} - a_{\alpha_k}) = a \in A$$

an arbitrary element of the cross cut $A \cap B$. To this element there corresponds a relation (3), which is, by hypothesis, satisfied by $x_{\alpha_1} = a_{\alpha_1}, \ldots, x_{\alpha_k} = a_{\alpha_k}$. Hence $a = 0$ and $A \cap B = 0$. This completes the proof of (4), and at the same time also that of Theorem 2.

Finally we remark that the above results admit an obvious generalization to modules with a non-commutative principal ideal ring as operator domain.

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[4]) See [3]).

Bibliography.

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