Infinite rings without infinite proper subrings.

To Professor László Kalmár on his 50th birthday.

By István Kovács in Debrecen.

§ 1. Introduction.

It is known that an infinite abelian group every proper subgroup of which is finite is isomorphic to Prüfer's quasicyclic group $C(p^\infty)$ [6].[1])[2]) The corresponding problem for non-commutative groups has not been solved hitherto. In the present note I give the solution of the analogous problem for rings (Theorem 2). It turns out that all infinite rings every proper subring of which is finite are given by certain subfields of the algebraic closure $\overline{P}_p$ of the prime field P_p of prime characteristic p and by the zero-rings[3]) with additive group $C(p^\infty)$. In particular any such ring is commutative and countable.

We get, moreover, the following result (Theorem 1): Every proper left ideal of an infinite ring R is finite if and only if R is a skew field or a zero-ring with additive group $C(p^\infty)$. [Of course the same result holds also for the case of right ideals.] The rings with additive group $C(p^\infty)$ from certain point of view looking trivial, the latter can be considered as a characterization of all infinite skew fields, just as the following theorem yields a characterization of all skew fields ([4]): a ring containing no proper left ideal other than 0 is necessarily a skew field or a zero-ring with p elements (p a prime). We make use of this theorem in the following section.

---

[1]) Numbers in brackets refer to the Bibliography at the end of this note.

[2]) Prüfer's quasicyclic group $C(p^\infty)$ is isomorphic to the additive group modulo 1 of all rational numbers with p-power denominators where p denotes a fixed prime number.

[3]) A ring R is called a zero-ring if $ab=0$ holds for any elements a, b of R. — We remark that the only ring with $C(p^\infty)$ as additive group is the zero-ring.

§ 2. The infinite rings every proper left ideal of which is finite.

In this section we prove the following

Theorem 1. *Let R be an arbitrary infinite ring. Then every proper left ideal of R is finite if and only if R is a skew field or a zero-ring with additive group $C(p^\infty)$.*

PROOF. Let R be an infinite ring every proper left ideal of which is finite. If L is an arbitrary proper left ideal of R, then any element $a \in R$ produces an endomorphism

$$x \to ax \qquad (x \in L)$$

of the additive group L^+ of L. So we have an antihomorphism of the ring R into the endomorphism ring of L^+. Since L is finite, also the endomorphism ring of L^+ is finite. Therefore the kernel of the antihomomorphism in question (being infinite) is equal to R. This means that $RL = 0$, i. e., all elements of L are right annihilators of R.

Now we form the union U of all proper left ideals in R. Thus U is a left ideal of R for which

$$RU = 0 \tag{1}$$

holds. If $U = R$, then R is a zero-ring, so that any proper additive subgroup of R must be finite. Hence, by [6], R is a zero-ring with additive group $C(p^\infty)$.

We suppose in the sequel that U is a proper (i. e. a finite) left ideal of R and we shall prove that in this case R is a skew field or, what is the same, that $U = 0$ (cf. the end of § 1).

First we remark that, by definition of U, R contains no proper left ideal containing U other than U. Therefore the set of all right annihilators of R is exactly U since this set is a two-sided ideal of R containing U (see (1)) and R. $R = 0$ conradicts our hypothesis that the union U of all proper left ideals in R is itself a proper left ideal of R. So we have obtained that U is a two-sided ideal in R such that the (infinite) factor ring R/U contains no proper left ideal other than 0. Hence R/U is a skew field. Let e be an element of R such that the residue class $e + U$ is just the unit element of the skew field R/U. Then Re is an infinite left ideal of R, i. e. $Re = R$ and

$$Ue = U. \tag{2}$$

On the other hand, since (1) implies $U^2 = 0$, we observe that each element of the factor ring R/U (i. e. each residue class of R modulo U) produces by right multiplication, a well defined endomorphism of the additive group of U. So we get a homomorphic representation of R/U by endomorphisms of the additive group of U. But R/U is an infinite skew field, U is finite, consequently this representation must be trivial: $Ua = 0$ for any $a \in R$. In parti-

cular $Ue=0$ which gives together with (2) $U=0$, completing so the proof of Theorem 1.

§ 3. The infinite rings every proper subring of which is finite.

We denote by $\bar{P}_p$ the algebraic closure of the prime field P_p of prime characteristic p. Let q be an arbitrary prime number (also the case $q=p$ is allowed). The field $\bar{P}_p$ contains exactly one subfield $GF(p^{q^k})$ with p^{q^k} elements ($k=0, 1, 2, \ldots$). We denote by $P_p(q^\infty)$ the union of all subfields $GF(p)$, $GF(p^q)$, $GF(p^{q^2})$, ... Obviously $P_p(q^\infty)$ is an infinite field all proper subrings of which are given by the finite fields $GF(p^{q^k})$ ($k=0, 1, 2, \ldots$) and by 0. Now we have the following

Theorem 2. *Every proper subring of an infinite ring R is finite if and only if R is a field of type $P_p(q^\infty)$ or a zero-ring with additive group $C(p^\infty)$.*

PROOF. Let R be an infinite ring every proper subring of which is finite. By Theorem 1. we have only to show that if R is a skew field, then R is isomorphic to a field $P_p(q^\infty)$.

First we observe that the skew field R has a characteristic $p>0$, since the rational number field has infinite proper subrings. We get, moreover, that every element of R is algebraic over the prime field P_p of R. In fact, if $x\in R$ would be transcendental over P_p, then the polynomial ring $P_p[x]$ would be an infinite proper subring of R. Thus, by a theorem of JACOBSON,[4]) R is commutative, i. e., R is a subfield of the algebraic closure $\bar{P}_p$ of P_p.

For an arbitrary subfield R of $\bar{P}_p$ we denote by $S(R)$ the set of all positive integers m such that R contains a subfield $GF(p^m)$ with p^m elements. Then the subfield R of $\bar{P}_p$ is uniquely determined by the set $S(R)$, and for a given set S of positive integers there exists a subfield R of $\bar{P}_p$ with $S(R)=S$ if and only if S contains for each $m\in S$ all (positive) divisors of m and for each pair $m\in S, m'\in S$ the least common multiple of m, m' ([3], § 16). Now we have an infinite subfield R of $\bar{P}_p$. Therefore also the set $S(R)$ is infinite. Suppose that $S(R)$ contains an infinity of distinct prime numbers $q_1, q_2, q_3, \ldots$. Then the subfield R' of R belonging to the set $S(R')$ which one obtains from $S(R)$ by deleting all element divisible by q_1 is an infinite proper subring of R. Since this is impossible we have obtained that $S(R)$ contains only a finite number $q_1, \ldots, q_n$ of distinct primes. But then the infinite set $S(R)$ contains for at least one q, say $q=q_1$, elements divisible by q^k with arbitrarily large values of k. This means that the infinite subfield R'' of $\bar{P}_p$ with

$$S(R'')=\{q, q^2, q^3, \ldots\}$$

[4]) See Theorem 2 on p. 701 in [2]. For a simple elementary proof of this theorem see [1].

is a subfield of R, i. e. $R = R'' = P_p(q^\infty)$, completing so the proof of Theorem 2.

REMARKS. Theorem 2 implies the following result: *if R is a ring every proper subring of which possesses at most N elements* (where N is a positive integer, fixed for a given R), *then R is finite.* This answers the analogue of a group-theoretic problem of G. GRÜNWALD.

Theorem 2 implies also the following theorem of T. SZELE [5]: *If a ring R satisfies both chain conditions for subrings, then R is finite.* As a matter of fact, suppose there exists an infinite ring R satisfying both chain conditions for subrings. Then a suitable infinite subring R' of R contains no infinite proper subring. But this contradicts our Theorem 2 since no ring covered by Theorem 2 satisfies the ascending chain condition for subrings.

ADDED IN PROOF (March 8, 1955): Professor A. G. KUROŠ has kindly directed my attention to the fact that our Theorem 2 follows also from a more general result of V. I. SNEJDMYULLER [*Mat. Sbornik N. S.* **27** (69) (1950), 219—228]. — It may be supposed, however, that the present proof retains some interest, being short and straighforward.

Bibliography.

[1] I. N. HERSTEIN, An elementary proof of a theorem of Jacobson. *Duke Math. J.*, **21** (1954), 45—48.

[2] N. JACOBSON, Structure theory for algebraic algebras of bounded degree. *Ann. of Math.*, (Princeton) (2), **46** (1945), 695—707.

[3] E. STEINITZ, Algebraische Theorie der Körper. *J. f. d. reine u. angew. Math.* **137** (1910). 167—308.

[4] T. SZELE, Die Ringe ohne Linksideale. *Buletin Stiintific*, **1** (1950), 788—789.

[5] T. SZELE, A finiteness criterion for rings. *Publ. Math. Debrecen*, **3** (1954), 58—60.

[6] I. SZÉLPÁL, Die unendlichen Abelschen Gruppen mit lauter endlichen echten Untergruppen. *Publ. Math. Debrecen*, **1** (1949), 63—64.

(Received December 31, 1954.)