Publ. Math. Debrecen
47 / 3-4 (1995), 403–410

Sequences of connected spectrum and the Vilenkin group

By ZOLTÁN BOROS (Debrecen)

Abstract. The author presents a characterization of the sequences that satisfy a generalization of the interval-filling property. In the second part an application in harmonic analysis is given.

1. Notation and general results

Let $\mathbb{K}$ denote the field of real or complex numbers throughout this section. When a linear normed space X is in consideration, put

$$\ell_1(X) = \left\{ (b_n) : \mathbb{N} \to X \;\middle|\; \sum_{n=1}^{\infty} \|b_n\| < \infty \right\}.$$

Definition. The Cartesian product $P = \prod_{n=1}^{\infty} P_n$ is called a *coefficient system in* $\mathbb{K}$ if P_n is a non-void, finite subset of $\mathbb{K}$ for every $n \in \mathbb{N}$. The coefficient system P is *bounded* if there exists $K \in \mathbb{R}$ such that $|p| \leq K$ for every $p \in \bigcup_{n=1}^{\infty} P_n$.

When X is a Banach space over $\mathbb{K}$, $P = \prod_{n=1}^{\infty} P_n$ is a bounded coefficient system in $\mathbb{K}$ and $b = (b_n) \in \ell_1(X)$ set

$$\|P\| = \sup\{\, |p| : p \in P_n \text{ for some } n \in \mathbb{N}\}$$

$$S_n(P,b) = \left\{ \sum_{k=1}^{n} \delta_k b_k \;\middle|\; \delta_k \in P_k \text{ for } k = 1,2,\dots,n \right\} \quad (n \in \mathbb{N}) \quad \text{and}$$

Mathematics Subject Classification: Primary 11B05; Secondary 43A32.
Key words and phrases: interval-filling sequences, Vilenkin group.
Research supported by the Hungarian National Scientific Research Foundation, Operating Grant Number OTKA 1652.

$$S(P,b)=\left\{\sum_{n=1}^{\infty}\delta_n b_n \;\middle|\; \delta_n\in P_n \text{ for every } n\in\mathbb{N}\right\}.$$

Definition. The set $S(P,b)$ is called the *P-spectrum of* (b_n).

We wish to characterize the sequences (b_n) the P-spectrum of which is connected with respect to a given coefficient system P. In case $X=\mathbb{K}=\mathbb{R}$ these sequences are called interval-filling (of type P) and discussed in [1], [2]. Though the complete characterization of sequences with connected P-spectrum, given in Theorem 1.3, seems to be rather complicated, all the known results [1] for $X=\mathbb{R}$ and some new results for $X=\mathbb{C}$ (with specified coefficient systems, cf. Section 2) can be directly derived from it.

First let us draw up a simple remark, which makes further argument more convenient.

Lemma 1.1. *Let $P=\prod_{n=1}^{\infty}P_n$ be a bounded coefficient system in $\mathbb{K}$, $p_n^*\in P_n$ and $P_n^\circ=\{p-p_n^*\mid p\in P_n\}$ for $n\in\mathbb{N}$, $P^\circ=\prod_{n=1}^{\infty}P_n^\circ$, X a Banach space over $\mathbb{K}$ and $b=(b_n)\in\ell_1(X)$. Then $S(P^\circ,b)$ is connected if and only if $S(P,b)$ is connected.*

Proof. Observe that $x\in S(P^\circ,b)$ if and only if $x+\sum_{n=1}^{\infty}p_n^*b_n\in S(P,b)$, thus the P°-spectrum and the P-spectrum of (b_n) are congruent.

The following theorem, which is proved for special cases in [3] and [4], plays a fundamental role in our investigations.

Theorem 1.1. *If X is a Banach space over $\mathbb{K}$, $P=\prod_{n=1}^{\infty}P_n$ is a bounded coefficient system in $\mathbb{K}$ and $b=(b_n)\in\ell_1(X)$, then the set $S(P,b)$ is compact.*

Proof. For $\delta=(\delta_n)\in P$ define $\phi_n(\delta)=\delta_n b_n$ $(n\in\mathbb{N})$. Consider the discrete topology on P_n and the product topology on P. The finite sets P_n are compact, thus P is also compact. Since $\phi_n:P\to X$ is a composition of a projection and a multiplication, it is continuous. The sum $\phi=\sum_{n=1}^{\infty}\phi_n$ is uniformly convergent, consequently $\phi:P\to X$ is continuous, hence its range $\phi(P)=S(P,b)$ is compact.

To formulate the following results we need further notation. When (Y,ϱ) is a metric space and $A\subset Y$ is finite, denote by $d(A)$ the diameter of A; for $\varepsilon>0$ define

$$T_\varepsilon(A)=\{(a_1,a_2)\in A\times A\mid \varrho(a_1,a_2)\le\varepsilon\}\qquad\text{and}$$

$$r(A)=\inf\left\{\varepsilon\in\,]0,\infty[\;\middle|\;\bigcup_{k=1}^{\infty}T_\varepsilon^k(A)=A\times A\right\}$$

(where the power means repeated composition). Due to the finiteness of A the above set is non-void and $r(A)$ is its minimum, moreover there exists $m \in \mathbb{N}$ such that $T^m_{r(A)}(A) = A \times A$ and $m \leq \operatorname{card}(A) - 1$.

Theorem 1.2. *If X is a Banach space over $\mathbb{K}$, $P = \prod_{n=1}^{\infty} P_n$ is a bounded coefficient system in $\mathbb{K}$, $b = (b_n) \in \ell_1(X)$ and $\liminf_{n\to\infty} r(S_n(P,b)) = 0$, then the P-spectrum of (b_n) is connected.*

Proof. Due to Lemma 1.1 we may assume that $0 \in P_n$ for every $n \in \mathbb{N}$. Then

$$S_n(P,b) \subset S_{n+1}(P,b) \subset S(P,b) \tag{1}$$

holds for every natural number n. To give an indirect proof to our theorem suppose that there exist non-void, disjoint, closed subsets E and F of $S(P,b)$ with $E \cup F = S(P,b)$. Then E and F are compact sets (cf. Theorem 1.1), thus their distance $\varrho(E,F)$ is positive. Hence there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have

$$\|P\| \sum_{k=n+1}^{\infty} \|b_k\| < \varrho(E,F). \tag{2}$$

Now fix a natural number $n > n_0$ and choose $x \in E \cap S_n(P,b)$. If $\delta_k \in P_k$ for $k \geq n+1$, then (2) implies $x + \sum_{k=n+1}^{\infty} \delta_k b_k \notin F$. From this and (1) it follows that the nth partial sum of any representation of an arbitrary element of F is in $F \cap S_n(P,b)$, so $F \cap S_n(P,b)$ and similarly $E \cap S_n(P,b)$ are non-void sets. For $0 < \varepsilon < \varrho(E,F)$ we obviously have

$$T_\varepsilon(S_n(P,b)) \subset (E \cap S_n(P,b))^2 \cup (F \cap S_n(P,b))^2 \subsetneqq (S_n(P,b))^2,$$

which implies the same property for $T^k_\varepsilon(S_n(P,b))$ whenever $k \in \mathbb{N}$. Therefore $r(S_n(P,b)) \geq \varrho(E,F)$, hence $\liminf r(S_n(P,b)) \geq \varrho(E,F) > 0$ in contradiction with the hypothesis.

The previous theorem is convenient for applications (cf. Section 2), while the complete characterization of sequences with connected P-spectrum is given in the following

Theorem 1.3. *Let X be a Banach space over $\mathbb{K}$, $P = \prod_{n=1}^{\infty} P_n$ a bounded coefficient system in $\mathbb{K}$ and $b = (b_n) \in \ell_1(X)$. The P-spectrum of (b_n) is connected if and only if*

$$r(S_n(P,b)) \leq \sum_{k=n+1}^{\infty} d(P_k)\|b_k\| \tag{3}$$

holds for every natural number n.

Proof. The sufficiency of the inequality-system (3) is an immediate consequence of Theorem 1.2. To prove its necessity we use an indirect reasoning again. Let us suppose that (3) does not hold for some $n \in \mathbb{N}$. Set $\sigma = \sum_{k=n+1}^{\infty} d(P_k)\|b_k\|$ and choose $x \in S_n(P,b)$ arbitrary. Observe that the assumptions $G_i \subset S_n(P,b)$, $x \in G_i$ and $r(G_i) \le \sigma \quad (i = 1, 2, \dots, m)$ imply $r(\bigcup_{i=1}^{m} G_i) \le \sigma$, hence there exists a maximal set G_x with the assumed three properties of the sets G_i. The set $S_n(P,b) \setminus G_x$ is non-void, since (3) is false for n as we have supposed. If $y \in G_x$ and $z \in S_n(P,b) \setminus G_x$, then $\|y - z\| > \sigma$, therefore $y + \sum_{k=n+1}^{\infty} \delta_k b_k \ne z + \sum_{k=n+1}^{\infty} \epsilon_k b_k$ whenever $\delta_k,\ \epsilon_k \in P_k \quad (k = n+1, n+2, \dots)$. Let us now define

$$H_x = \left\{ y + \sum_{k=1}^{\infty} \delta_k b_{n+k} \;\middle|\; y \in G_x \text{ and } \delta_k \in P_k \text{ for every } k \in \mathbb{N} \right\}.$$

H_x and $S(P,b) \setminus H_x$ are non-void subsets of $S(P,b)$. H_x is a union of finitely many compact sets, hence it is compact. The same holds for its complement, since

$$S(P,b) \setminus H_x = \left\{ z + \sum_{k=1}^{\infty} \delta_k b_{n+k} \;\middle|\; \delta_k \in P_k \ (k = 1, 2, \dots),\ z \in S_n(P,b) \setminus G_x \right\}.$$

Therefore both of them is a closed set, thus $S(P,b)$ is disconnected in contradiction with the hypothesis.

2. Some Fourier-type transforms of the Vilenkin group

An analogue of Theorem 1 in [4] is given in this section. Consider a sequence $m = (m_n) : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ and let us denote by Z_s the discrete cyclic (additive) group of s elements (i.e. the set $\{0, 1, \dots, s-1\}$ with the modulo s addition). The Cartesian product (both in topological and algebraic sense)

$$G_m = \prod_{n=1}^{\infty} Z_{m_n}$$

is called the *Vilenkin group* [5, pages 501–510.]. For $n \in \mathbb{N}$ and $x = (x_1, x_2, \dots) \in G_m$ define

$$\varrho_n(x) = \exp\left(\frac{x_n}{m_n} 2\pi i \right).$$

The function ϱ_n is a character of G_m, it is called the (nth) Rademacher character of G_m. Any character of the Vilenkin group is a product of finitely many Rademacher characters. For a given sequence $a = (a_n) \in \ell_1(\mathbb{C})$ set

$$\phi_a = \sum_{n=1}^{\infty} a_n \varrho_n.$$

The sum is absolutely and uniformly convergent, thus $\phi_a : G_m \to \mathbb{C}$ is continuous. To illuminate the connection with the previous section, put

$$E_n = \left\{ \exp\left(\frac{k}{m_n} 2\pi i\right) \;\Big|\; k = 0, 1, \ldots, m_n - 1 \right\} \qquad (n \in \mathbb{N})$$

and $E = \prod_{n=1}^{\infty} E_n$. Observe that E is a bounded coefficient system in $\mathbb{C}$ and $\phi_a(G_m) = S(E, a)$.

Lemma 2.1. *If* $u,\ v,\ w$ *are complex numbers and*

$$\left| \arg\left(\frac{w-u}{v-u}\right) \right| \le \frac{\pi}{6},$$

then

$$|w - v| \le \max\left\{ |w-u|\ ,\ |v-u| - \frac{1}{\sqrt{3}} |w-u| \right\}$$

(we consider the function $\arg : \mathbb{C} \setminus \{0\} \to]-\pi, \pi]$ *here).*

Proof. Set

$$\alpha = \left| \arg\left(\frac{w-u}{v-u}\right) \right| \quad \text{and} \quad \beta = \left| \arg\left(\frac{w-v}{u-v}\right) \right|.$$

Consider the case $|w - v| > |w - u|$, then $\beta \le \alpha$,

$$|w - v| = |v - u| \frac{\sin\alpha}{\sin(\alpha+\beta)} \quad \text{and} \quad |w - u| = |v - u| \frac{\sin\beta}{\sin(\alpha+\beta)},$$

as known from elementary geometry. To obtain the inequality in consideration, it is sufficient to justify

$$\sin(\alpha+\beta) - \sin\alpha - \frac{1}{\sqrt{3}} \sin\beta \ge 0 \quad \left(0 \le \beta \le \alpha \le \frac{\pi}{6}\right),$$

which can be performed by standard analysis of extreme values.

Theorem 2.1. *Let $a = (a_n) \in \ell_1(\mathbb{C})$ and G_m the Vilenkin group. If $m_n \geq 6$ and*

$$|a_n| \leq \sum_{k=n+1}^{\infty} |a_k| \tag{4}$$

hold for every natural number n, then the set $\phi_a(G_m)$ is connected.

Proof. Due to Theorem 1.2 (or Theorem 1.3, since $1 < d(E_n)$) it suffices to show that

$$r(S_n(E,a)) \leq \sum_{k=n+1}^{\infty} |a_k| \tag{5}$$

is satisfied for every $n \in \mathbb{N}$. Set $\beta_n = \sum_{k=n+1}^{\infty} |a_k|$ $(n \in \mathbb{N})$ and let us prove (5) by induction. For $n = 1$ we have

$$r(S_1(E,a)) = 2|a_1| \sin \frac{\pi}{m_1} \leq 2|a_1| \sin \frac{\pi}{6} = |a_1| \leq \beta_1.$$

Assume that $n > 1$ and $r(S_{n-1}(E,a)) \leq \beta_{n-1}$. For $x \in S_{n-1}(E,a)$ put

$$A_x = \left\{ x + a_n \exp\left(\frac{k}{m_n} 2\pi i\right) \,\middle|\, k = 0, 1, \ldots, m_n - 1 \right\}.$$

Similarly, as it is calculated for the case $n = 1$, $r(A_x) \leq |a_n| \leq \beta_n$. Hence

$$A_x \times A_x \subset T_{\beta_n}^{m_n - 1}(S_n(E,a)).$$

Choose $x,\ y \in S_{n-1}(E,a)$ with $|x - y| \leq \beta_{n-1}$. We are going to prove that there exist $z \in A_x$ and $w \in A_y$ such that $|z - w| \leq \beta_n$. Then we can infer

$$(S_n(E,a))^2 = \left(\bigcup_{x \in S_{n-1}(E,a)} A_x \right)^2 = T_{\beta_n}^{t(m_n - 1) + t - 1}(S_n(E,a))$$

where $t = \operatorname{card}(S_{n-1}(E,a))$, that is $r(S_n(E,a)) \leq \beta_n$. The existence of such z and w is obvious when $|x - y| \leq \beta_n$ is satisfied: let $z = x + a_n$ and $w = y + a_n$. Otherwise choose $k \in \{0, 1, \ldots, m_n - 1\}$ such that

$$\left| \arg(y - x) - \arg(a_n) - \left(\frac{k}{m_n} + j \right) 2\pi \right| \leq \frac{\pi}{6}$$

hold for some $j \in \mathbb{Z}$ and let $z = x + a_n \exp\left(\frac{k}{m_n} 2\pi i\right)$. Lemma 2.1 implies

$$|z - y| \leq \max\left\{ |a_n| \,,\ |x - y| - \frac{|a_n|}{\sqrt{3}} \right\} \leq \max\left\{ |a_n| \,,\ |x - y| - \frac{|a_n|}{2} \right\}.$$

Now choose $l \in \{0, 1, \dots, m_n - 1\}$ such that

$$\left|\arg(z-y) - \arg(a_n) - \left(\frac{l}{m_n} + j\right)2\pi\right| \le \frac{\pi}{6}$$

hold for some $j \in \mathbb{Z}$ and let $w = y + a_n \exp\bigl(\frac{l}{m_n} 2\pi i\bigr)$. Applying Lemma 2.1 we have

$$\begin{aligned}
|z-w| &\le \max\left\{|a_n|\ ,\ |z-y| - \frac{|a_n|}{\sqrt{3}}\right\} \le \max\left\{|a_n|\ ,\ |z-y| - \frac{|a_n|}{2}\right\} \\
&\le \max\{|a_n|\ ,\ |x-y| - |a_n|\} \le \max\{|a_n|\ ,\ \beta_{n-1} - |a_n|\} \\
&= \max\{|a_n|\ ,\ \beta_n\} = \beta_n.
\end{aligned}$$

Theorem 2.2. *Let $a = (a_n) \in \ell_1(\mathbb{C})$ and G_m the Vilenkin group. If*

$$|a_n| \sin \frac{\pi}{m_n} > \sum_{k=n+1}^{\infty} |a_k| \tag{6}$$

holds for every natural number n, then the set $\phi_a(G_m)$ is totally disconnected.

Proof. Since G_m is a product of finite, discrete sets, it is compact and totally disconnected. We have also proved that the mapping $\phi_a : G_m \to \mathbb{C}$ is continuous, now we wish to justify that it is injective as well. It implies, as it is familiar, that ϕ_a is a homeomorphism. To prove the injectivity suppose that $x,\ y \in G_m,\ x \ne y$ and $\phi_a(x) = \phi_a(y)$. Put $p = \inf\{n \in \mathbb{N} \mid x_n \ne y_n\}$. Then the assumption $\phi_a(x) - \phi_a(y) = 0$ can be written in the form

$$a_p(\varrho_p(y) - \varrho_p(x)) = \sum_{n=p+1}^{\infty} a_n(\varrho_n(x) - \varrho_n(y)),$$

hence

$$\begin{gathered}
2|a_p| \sin \frac{\pi}{m_p} \le |a_p|\,|\varrho_p(y) - \varrho_p(x)| \le \\
\sum_{n=p+1}^{\infty} |a_n|\,|\varrho_n(x) - \varrho_n(y)| \le 2 \sum_{n=p+1}^{\infty} |a_n|,
\end{gathered}$$

in contradiction with the hypothesis.

References

[1] Z. Boros, Interval-filling sequences with respect to a finite set of real coefficients, *Publ. Math. Debrecen* **43** 1-2 (1993), 61–68.
[2] Z. Daróczy, A. Járai and I. Kátai, Intervalfüllende Folgen und volladditive Funktionen, *Acta Sci. Math.* **52** (1988), 337–350.
[3] Z. Daróczy and I. Kátai, Generalized number systems in the complex plane, *Acta Math. Hungar.* **51** 3-4 (1988), 409–416.
[4] Gy. Maksa, Interval-filling sequences and the dyadic group, *Grazer Math. Ber.* **315** (1991), 69–74.
[5] F. Schipp, P. Simon and W. R. Wade, Walsh series, *Akadémiai Kiadó*, *Budapest*, 1990.

ZOLTÁN BOROS
INSTITUTE OF MATHEMATICS AND INFORMATICS
LAJOS KOSSUTH UNIVERSITY
H–4010 DEBRECEN, PF. 12
HUNGARY
E-MAIL: boros@math.klte.hu

(Received March 1, 1995)