

On permutations of set products.

To the memory of my beloved teacher Professor Tibor Szele.

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1. Introduction. The set product $S = \prod_{t \in T} A_t$ is defined as the set of the functions $f(t)$ ($t \in T$) satisfying $f(t) \in A_t$ for each $t \in T$. We suppose that $A_t \cap A_{t'} = \bigcap_{t \in T} A_t$ for any $t, t' \in T$, $t \neq t'$, and that this intersection contains exactly one element e . In this case, one can give also an inner construction of S in the following way. Let us identify each element a of $A = \bigcup_{t \in T} A_t$ with the function $f \in S$ defined by

$$f(u) = \begin{cases} a & \text{if } u = t \quad (a \in A_t) \\ e & \text{if } u \neq t, \quad u \in T, \end{cases}$$

in particular, the element e with the function f_0 for which $f_0(u) = e$ for any $u \in T$. Then the set A_t becomes identical with the subset S_t of S consisting of those $f \in S$ for which $f(u) = e$ if $u \neq t$ ($t, u \in T$). Thus we can regard S as the product of its own subsets S_t ($t \in T$).

Now let be given an arbitrary group G of permutations of S satisfying the following conditions. Firstly, we require $ea = e$ for any permutation $a \in G$. Secondly, let be given to each $t \in T$ an $\alpha_t \in G$ such that $S_t \alpha_t = S_{t'}$ for suitable $t' \in T$ and $t \rightarrow t'$ is a permutation β of T . Then we require that the permutation α assigning to any $f \in S$ the $f' \in S$ defined by $f'(t') = f(t) \alpha_t$ ($t' \in T$, $t = t' \beta^{-1}$) be an element of G (the above identification makes clear meaning of the expression $f(t) \alpha_t$). We note that for any $t \in T$ and $f \in S_t$ we have $f\alpha = f\alpha_t$.

We shall define the following three subgroups $G_1 \supseteq G_2 \supseteq G_3$ of G . The permutation $\alpha \in G$ belongs to G_1 if and only if there exists for each $t \in T$ an index $t' \in T$ such that $S_t \alpha = S_{t'}$; $\alpha \in G_2$ if and only if $S_t \alpha = S_t$ by any $t \in T$; $\alpha \in G_3$ if and only if $x\alpha = x$ for every $x \in \bigcup_{t \in T} S_t$.

Our results are concerned with the group theoretical building up of the abstract groups G_i ($i = 1, 2, 3$) defined above.

2. Results.

THEOREM 1. G_1 is a splitting Schreier extension of G_2 by an unrestricted direct product of symmetric groups.

PROOF. For any $\alpha \in G_1$ the transformation $t \rightarrow t'$ defined by $S_t \alpha = S_{t'}$ is clearly a permutation of T ; we call it the permutation induced by α . Calling two indices t and t' ($\in T$) equivalent if and only if $S_t \alpha = S_{t'}$ for an $\alpha \in G_1$, the set T splits into equivalency classes. It is evident by the condition imposed on G that any permutation $\bar{\alpha}$ of T such that for any $t \in T$, $t\bar{\alpha}$ and t belong to the same class, is induced by some $\alpha \in G_1$ and so the group $\bar{G}_1$ of the permutations induced by the elements of G_1 can be decomposed into an unrestricted direct product of symmetric groups, viz. of those of the equivalency classes. Moreover the mapping $\alpha \rightarrow \bar{\alpha}$ is a homomorphism of the group G_1 onto $\bar{G}_1$ with kernel G_2 . Therefore G_2 is a normal subgroup of G_1 and G_1/G_2 is an unrestricted direct product of symmetric groups. Now our aim is to construct a subgroup H_1 of G_1 for which $G_1 = G_2 H_1$ and $G_2 \cap H_1 = 1$ hold. For this purpose let us consider a complete system T^* of representatives of the equivalency classes of the t 's. For any $t^* \in T^*$ and arbitrary $t \in T$, $t \neq t^*$, equivalent to t^* , let us choose an $\alpha \in G$ such that $S_{t^*} \alpha = S_t$ and denote with $\varphi_{t^*, t}$ the mapping α restricted to S_{t^*} , i. e. the mapping defined by $f\varphi_{t^*, t} = f\alpha$ for $f \in S_{t^*}$; moreover, denote φ_{t, t^*} the inverse of $\varphi_{t^*, t}$. Finally, let φ_{t^*, t^*} be the identical mapping of S_{t^*} onto itself. Furthermore, for any two equivalent indices t_1 and t_2 , we define φ_{t_1, t_2} by $\varphi_{t_1, t_2} = \varphi_{t_1, t^*} \varphi_{t^*, t_2}$ where $t^* \in T^*$ is the index belonging to the class containing both t_1 and t_2 . Any $\bar{\alpha} \in \bar{G}_1$, hence any coset of G_1 modulo G_2 determines uniquely a permutation $f \rightarrow f\alpha^*$ ($f \in S$, $\alpha^* \in G_1$), defined by $f\alpha^*(t\bar{\alpha}) = f(t)\varphi_{t, t\bar{\alpha}}$ ($t \in T$) such that the permutation induced by α^* is identical with $\bar{\alpha}$. The choice of the φ_{t_1, t_2} 's ensures that the permutations α^* form a subgroup H_1 of G_1 . Thus we have a subgroup $H_1 \subseteq G_1$ with the required properties.

For any $\beta \in G_2$ and $f \in S_t$, we have $f\beta = f' \in S_t$ and $f \rightarrow f'$ is clearly a permutation of S_t . We call now $f \rightarrow f'$ the permutation of S_t induced by β . Obviously, these permutations form a group G_t for any $t \in T$. Then we have

THEOREM 2. G_2 is a splitting Schreier extension of G_3 by the unrestricted direct product P of the groups of permutations G_t ($t \in T$).

PROOF. It is obvious that we have a homomorphism of G_2 onto P if we put into correspondence to each $\beta \in G_2$ the element $p \in P$ mapping each $t \in T$ onto the permutation of S_t induced by β ; the kernel of this homomorphism

is G_3 . Now, let be given for any $t \in T$ an element β_t of G_t . Define the permutation $f \rightarrow f\beta^*$ of S by $f\beta^*(t) = f(t)\beta_t$ ($t \in T$). These permutations β^* form a subgroup H_2 of G_2 , isomorphic to P , for which $G_2 = G_3 H_2$ and $G_3 \cap H_2 = 1$ hold.

COROLLARY 1. G_1 is a splitting Schreier extension of G_3 and G_1/G_3 is also a splitting Schreier extension of G_2/G_3 .

COROLLARY 2. Let S be an algebraic structure¹⁾ with exactly one substructure consisting of a single element of S . Let S be the unrestricted direct product of certain of its substructures S_t ($t \in T$). Let G be the automorphism group of S . Then G_1 is a splitting Schreier extension of G_2 by an unrestricted direct product of symmetric groups and G_2 is a splitting Schreier extension of G_3 by the unrestricted direct product of the automorphism groups of the S_t 's ($t \in T$).

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¹⁾ For the definition of algebraic structure see e. g. G. BIRKHOFF: Lattice theory, New York, 1948.