

Note on orthogonal polynomials related to theta functions.

By L. CARLITZ in Durham, North Carolina.

1. Put

$$(1.1) \quad H_n(x) = H_n(x, q) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} x^r,$$

$$(1.2) \quad G_n(x) = G_n(x, q) = \sum_{r=0}^n \begin{bmatrix} n \\ r \end{bmatrix} q^{r(r-n)} x^r,$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q^{n-r+1})}{(1-q)(1-q^2) \cdots (1-q^r)}, \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = 1.$$

SZEGŐ [2] showed that

$$(1.3) \quad \frac{1}{2\pi} \int_0^{2\pi} H_m(-q^{-\frac{1}{2}} e^{i\psi}) H_n(-q^{-\frac{1}{2}} e^{-i\psi}) f(\psi) d\psi = q^{-n} (q)_n \delta_{mn},$$

where

$$(1.4) \quad f(\psi) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} e^{ni\psi} = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} \cos n\psi \quad (|q| < 1)$$

and

$$(q)_n = (1-q)(1-q^2) \cdots (1-q^n), \quad (q)_0 = 1.$$

WIGERT [3] proved that

$$(1.5) \quad \int_0^{\infty} G_m(-q^{n+\frac{1}{2}} x) G_n(-q^{n+\frac{1}{2}} x) p(x) dx = q^{-n-\frac{1}{2}} (q)_n \delta_{mn},$$

where

$$(1.6) \quad p(x) = k\tau^{-\frac{1}{2}} \exp(-k^2 \log^2 x),$$

and

$$(1.7) \quad 2k^2 = -\frac{1}{\log q} \quad (0 < q < 1).$$

For some additional properties of $H_n(x)$ and $G_n(x)$ see [1].

The object of the present note is to prove (1.3) and (1.5) as well as some related results in a uniform and somewhat simpler way.

2. We first prove a preliminary result. Put

$$(2.1) \quad a_{mn} = \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1) - rs}$$

Then

$$(2.2) \quad a_{mn} = q^{-n} (q)_n \delta_{mn}.$$

We recall the familiar identity

$$(2.3) \quad \prod_{s=0}^{n-1} (1 - q^s x) = \sum_{s=0}^n (-1)^s \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}s(s-1)} x^s.$$

Thus (2.1) becomes

$$(2.4) \quad a_{mn} = \sum_{r=0}^m (-1)^r \begin{bmatrix} m \\ r \end{bmatrix} q^{\frac{1}{2}r(r-1)} \prod_{s=0}^{n-1} (1 - q^{s-r}).$$

There is no loss in generality in assuming $m \leq n$. For $m < n$ it is evident that the product in the right member of (2.4) will vanish for all r ; hence

$$(2.5) \quad a_{mn} = 0 \quad (m \neq n).$$

For $m = n$, on the other hand, the term $r = m$ yields

$$(2.6) \quad a_{nn} = (-1)^n q^{\frac{1}{2}n(n-1)} \prod_{s=0}^{n-1} (1 - q^{s-n}) = q^{-n} \prod_{s=1}^n (1 - q^s) = q^{-n} (q)_n.$$

Combining (2.5) and (2.6), we have proved (2.2).

If in (2.1) we replace q by q^{-1} we find that

$$(2.7) \quad \begin{aligned} a'_{mn} &= \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r+1) + \frac{1}{2}s(s+1) - rm - sn + rs} = \\ &= (-1)^n q^{-\frac{1}{2}n(n-1)} (q)_n \delta_{mn}. \end{aligned}$$

Other variants are obtained by replacing r by $m-r$ or s by $n-s$ in (2.1).

We get

$$(2.8) \quad \begin{aligned} b_{mn} &= \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r+1) + \frac{1}{2}s(s-1) - m(r+s) + rs} = \\ &= (-1)^n q^{-\frac{1}{2}n(n+1)} (q)_n \delta_{mn}, \end{aligned}$$

$$(2.9) \quad \begin{aligned} c_{mn} &= \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s+1) - n(r+s) + rs} = \\ &= (-1)^n q^{-\frac{1}{2}n(n+1)} (q)_n \delta_{mn}, \end{aligned}$$

$$(2.10) \quad \begin{aligned} d_{mn} &= \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r+1) + \frac{1}{2}s(s+1) - (m-n)(r-s) - rs} = \\ &= (q)_n \delta_{mn}. \end{aligned}$$

3. To prove (1.3) we use (1.1) and (1.4). Then

$$\begin{aligned}
 & \int_0^{2\pi} H_m(-q^{-\frac{1}{2}} e^{i\psi}) H_n(-q^{-\frac{1}{2}} e^{-i\psi}) f(\psi) d\psi = \\
 &= \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{-\frac{1}{2}(r+s)} \sum_{t=-\infty}^{\infty} q^{\frac{1}{2}t^2} \int_0^{2\pi} e^{(r-s+t)i\psi} d\psi = \\
 &= 2\pi \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{-\frac{1}{2}(r+1) + \frac{1}{2}(s-1)^2} = \\
 &= 2\pi \sum_{r=0}^m \sum_{s=0}^n (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}(r-1) + \frac{1}{2}s(s-1) - rs} = \\
 &= 2\pi a_{mn} = 2\pi q^{-n} (q)_n \delta_{mn},
 \end{aligned}$$

where at the last step we use (2.1) and (2.2).

This evidently completes the proof of (1.3).

4. To prove (1.5) we require the following formula, which is easily verified:

$$(4.1) \quad \int_0^{\infty} x^n p(x) dx = q^{-\frac{1}{2}(n+1)^2}.$$

This formula holds for all integral n .

Using (1.2) and (4.1) we get

$$\begin{aligned}
 & \int_0^{\infty} G_n(-q^{m+\frac{1}{2}} x) G_n(-q^{n+\frac{1}{2}} x) p(x) dx = \\
 &= \sum_{r,s} (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{r^2+s^2 + \frac{1}{2}(r+s)} \int_0^{\infty} x^{r+s} p(x) dx = \\
 &= \sum_{r,s} (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{r^2+s^2 + \frac{1}{2}(r+s)} q^{-\frac{1}{2}(r+s+1)^2} = \\
 &= q^{-\frac{1}{2}} \sum_{r,s} (-1)^{r+s} \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ s \end{bmatrix} q^{\frac{1}{2}r(r-1) + \frac{1}{2}s(s-1) - rs} = \\
 &= q^{-\frac{1}{2}} a_{mn} = q^{-n-\frac{1}{2}} (q)_n \delta_{mn},
 \end{aligned}$$

which proves (1.5).

5. If in place of (2.2) we make use of one of the formulas (2.7) to (2.10), we obtain a number of results similar to (1.3) and (1.5). Thus by means of (2.7) we get

$$(5.1) \quad \frac{1}{2\pi} \int_0^{2\pi} H_m(-q^{-m+\frac{1}{2}} e^{i\psi}) H_n(-q^{-n+\frac{1}{2}} e^{i\psi}) f(\psi) d\psi = \\ = (-1)^n q^{-\frac{1}{2}n(n-1)} (q)_n \delta_{mn},$$

$$(5.2) \quad \int_0^\infty G_m(-x) G_n(-qx^{-1}) p(x) dx = (-1)^n q^{-\frac{1}{2}(n^2-n+1)} (q)_n \delta_{mn}.$$

Similarly use of (2.8) gives

$$(5.3) \quad \frac{1}{2\pi} \int_0^{2\pi} H_m(-q^{-m+\frac{1}{2}} e^{i\psi}) H_n(-q^{-n-\frac{1}{2}} e^{i\psi}) f(\psi) d\psi = \\ = (-1)^n q^{-\frac{1}{2}n(n+1)} (q)_n \delta_{mn},$$

$$(5.4) \quad \int_0^\infty G_m(-x) G_n(-x^{-1}) p(x) dx = (-1)^n q^{-\frac{1}{2}(n^2+n+1)} (q)_n \delta_{mn},$$

while (2.9) yields

$$(5.5) \quad \frac{1}{2\pi} \int_0^{2\pi} H_m(-q^{-n-\frac{1}{2}} e^{i\psi}) H_n(-q^{-n+\frac{1}{2}} e^{i\psi}) f(\psi) d\psi = \\ = (-1)^n q^{-\frac{1}{2}n(n+1)} (q)_n \delta_{mn},$$

$$(5.6) \quad \int_0^\infty G_m(-x^{-1}) G_n(-x) p(x) dx = (-1)^n q^{-\frac{1}{2}(n^2+n+1)} (q)_n \delta_{mn}.$$

We note that (5.3) and (5.5) are equivalent; also (5.4) and (5.6) are equivalent.

Finally by means of (2.10) we obtain

$$(5.7) \quad \frac{1}{2\pi} \int_0^{2\pi} H_m(-q^{n-m+\frac{1}{2}} e^{i\psi}) H_n(-q^{m-n+\frac{1}{2}} e^{-i\psi}) f(\psi) d\psi = (q)_n \delta_{mn}.$$

$$(5.8) \quad \int_0^\infty G_m(-q^{n+\frac{3}{2}} x) G_n(-q^{m+\frac{3}{2}} x) p(x) dx = q^{-\frac{1}{2}} (q)_n \delta_{mn}.$$

If in place of (1.4) we employ

$$(5.9) \quad f_1(\psi) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n-1)} e^{ni\psi},$$

it is easily verified that

$$(5.10) \quad \frac{1}{2\pi} \int_0^{2\pi} H_m(-e^{i\psi}) H_n(-q^{-1}e^{-ni\psi}) f_1(\psi) d\psi = q^{-n}(q)_n \delta_{mn}.$$

Similarly we can find variants of (5.1), (5.3) and (5.7). Variants of (1.5), (5.2), (5.4), (5.8) are obtained by using in place of (1.6) the weight function

$$p_1(x) = k\pi^{-\frac{1}{2}} \exp \left\{ - \left(k \log x + \frac{1}{4k} \right)^2 \right\} = kq^{\frac{1}{2}} (\pi x)^{-\frac{1}{2}} p(x)$$

which satisfies

$$\int_0^{\infty} x^n p_1(x) dx = q^{-\frac{1}{2}n(n+1)}$$

6. Since

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{(q)_n} = \prod_{n=0}^{\infty} (1 - q^n t)^{-1} (1 - q^n x t)^{-1} \quad (|t| < 1)$$

and

$$\sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} z^n = \prod_{n=1}^{\infty} (1 - q^n) (1 - q^{n-1} z) (1 - q^{n-1} z^{-1}),$$

it is easily verified that (1.3) implies

$$(6.1) \quad \frac{1}{2\pi} \int_0^{2\pi} \prod_{n=1}^{\infty} \frac{(1 - q^n e^{i\psi})(1 - q^n e^{-i\psi})}{(1 + q^{n+\frac{1}{2}} e^{i\psi} t)(1 + q^{n+\frac{1}{2}} e^{-i\psi} z)} d\psi = \\ = \prod_{n=1}^{\infty} \frac{(1 - q^n t)(1 - q^n z)}{(1 - q^n)(1 - q^n t z)}.$$

Similarly since

$$\sum_{n=0}^{\infty} (-1)^n q^{\frac{1}{2}n(n-1)} G_n(x) \frac{t^n}{(q)_n} = \prod_{n=0}^{\infty} (1 - q^n t) (1 - q^n x t),$$

(5.4) is equivalent to

$$(6.2) \quad \int_0^{\infty} \prod_{n=1}^{\infty} (1 + q^n x t) (1 + q^n x^{-1} z) \cdot p(x) dx = q^{-\frac{1}{2}} \prod_{n=1}^{\infty} \frac{1 - q^n t z}{(1 - q^n t)(1 - q^n z)}.$$

7. The following formulas are of a slightly different kind. For brevity put

$$(x)_n = \prod_{s=0}^{n-1} (1 - q^s x),$$

the left member of (2.3). Then we have

$$(7.1) \quad \frac{1}{2\pi} \int_0^{2\pi} H_n(xe^{i\psi}) f(\psi) d\psi = (-xq^{\frac{1}{2}})_n,$$

$$(7.2) \quad \frac{1}{2\pi} \int_0^{2\pi} (xe^{i\psi})_n f(\psi) d\psi = G_n(-q^{n-\frac{1}{2}}x),$$

$$(7.3) \quad \int_0^\infty G_n(xz) p(z) dz = q^{-\frac{1}{2}} (-xq^{-n-\frac{1}{2}})_n,$$

$$(7.3)' \quad \int_0^\infty G_n(xz) p_1(z) dz = (-xq^{-n})_n,$$

$$(7.4) \quad \int_0^\infty (xz)_n p(z) dz = q^{-\frac{1}{2}} H_n(-xq^{-\frac{3}{2}}),$$

$$(7.4)' \quad \int_0^\infty (xz)_n p_1(z) dz = H_n(-xq^{-1}),$$

where in (7.3)' and (7.4)', $p_1(z)$ has the same meaning as in (5.11).

Finally if

$$f(\psi, q^2) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{ni\psi},$$

$$p(x, q^2) = k(2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} k^2 \log^2 x\right),$$

where k is defined by (1.7), then we have also

$$(7.5) \quad \frac{1}{2\pi} \int_0^{2\pi} H_n(xe^{i\psi}) f(\psi, q^2) d\psi = G_n(xq^n),$$

$$(7.6) \quad \int_0^\infty G_n(xz) p(z, q^2) dz = q^{-1} H_n(xq^{-n-2}).$$

The proof of these formulas is immediate.

We remark that if

$$f(\psi, z, q^2) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{ni} e^{ni\psi},$$

then

$$(7.7) \quad \frac{1}{2\pi} \int_0^{2\pi} H_n(xe^{i\psi}) f(\psi, z, q^2) d\psi = G_n(xz^{-1}q^n).$$

The formulas (7.1) and (7.2) can be extended in a similar manner.

Bibliography.

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