

The conductor of a cyclic quartic field

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Abstract. Explicit formulae are obtained for the conductor and the discriminant of a cyclic quartic field $K = Q(\theta)$, where θ is a root of an irreducible polynomial $q(X) = X^4 + AX^2 + BX + C \in Z[X]$, and the integers A, B, C are such that there are no primes p with $p^2 \mid A$, $p^3 \mid B$, $p^4 \mid C$.

Let Z denote the domain of rational integers, let Q denote the field of rational numbers, and let K be a cyclic quartic extension field of Q , that is, $[K:Q] = 4$ and $\text{Gal}(K/Q) \simeq Z/4Z$. As K is a normal extension of Q and $\text{Gal}(K/Q)$ is an abelian group, K is an abelian field, and so by the Kronecker-Weber Theorem there exists a positive integer f such that $K \subseteq Q(\exp(2\pi i/f))$. The least such positive integer f is called the conductor of K and is denoted by $f(K)$. In this paper we take K in the form $K = Q(\theta)$, where θ is a root of an irreducible polynomial $q(X) = X^4 + AX^2 + BX + C \in Z[X]$, and determine $f(K)$ explicitly in terms of the coefficients A, B, C of $q(X)$. As $q(X)$ is irreducible over Z , we cannot have $A^2 - 4C = B = 0$. From [3] and [4] it is easy to deduce a necessary and sufficient condition for the splitting field K of the irreducible polynomial $q(X)$ to be cyclic.

For a prime p and a non-zero integer m , we denote by $v_p(m)$ the largest exponent k such that $p^k \mid m$, and write $p^{v_p(m)} \parallel m$. If for any prime p we have

$$v_p(A) \geq 2, \quad v_p(B) \geq 3, \quad v_p(C) \geq 4,$$

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then θ/p is an algebraic integer, which is a root of the irreducible polynomial

$$X^4 + (A/p^2)X^2 + (B/p^3)X + (C/p^4) \in Z[X],$$

and $K = Q(\theta/p)$. Therefore we can make the following simplifying assumption:

- (1) there does not exist a prime p such that $p^2 \mid A$, $p^3 \mid B$, $p^4 \mid C$.

Our main result is the following theorem.

Theorem 1. *Let $K = Q(\theta)$ be a cyclic quartic extension of Q , where θ is a root of the irreducible polynomial $q(X) = X^4 + AX^2 + BX + C \in Z[X]$ with coefficients A, B, C satisfying (1).*

Case (i): $A^2 - 4C \neq 0$, $B \neq 0$: Set

$$\ell = v_2(A^2 - 4C), \quad b = v_2(B),$$

and for a prime $p \neq 2$ set

$$e_p = \min(v_p(A^2 - 4C), v_p(B)).$$

Then

$$f(K) = 2^\alpha \prod_{\substack{p \neq 2 \\ e_p \text{ odd}}} p \prod_{\substack{p \neq 2 \\ e_p(\text{even}) \geq 2, p \mid A}} p,$$

where the values of α are given in TABLE (i).

Case (ii): $A^2 - 4C = 0$, $B \neq 0$: Here

$$f(K) = 2^\beta \prod_{\substack{p \neq 2 \\ v_p(B) \text{ odd}}} p \prod_{\substack{p \neq 2 \\ v_p(B)(\text{even}) \geq 2, p \mid A}} p,$$

where the values of β are given in TABLE (ii).

Case (iii): $A^2 - 4C \neq 0$, $B = 0$: Here

$$f(K) = 2^\gamma \prod_{\substack{p \neq 2 \\ p \mid A, p \mid C}} p,$$

where the values of γ are given in TABLE (iii).

PROOF of Theorem 1. We just treat case (i) ($A^2 - 4C \neq 0$, $B \neq 0$) as cases (ii) and (iii) can be treated in a similar but easier manner.

We begin by outlining the ideas involved in the proof. First we solve the quartic equation $q(\theta) = \theta^4 + A\theta^2 + B\theta + C = 0$ for θ in terms of

A, B, C and the unique integral root t of the cubic resolvent of $q(X)$, see (2) and (3). We then use this solution to express $K = Q(\theta)$ in the form $K = Q(\sqrt{m + n\sqrt{S}})$, where m, n, S are integers such that (m, n) and S are both squarefree and $m + n\sqrt{S}$ is not a square in $Q(\sqrt{S})$, see (11) and (12). Various relationships involving A, B, C, t, S, m, n are recorded in (4)–(10) for later use. For K expressed in the form $Q(\sqrt{m + n\sqrt{S}})$, Huard, Spearman and Williams have given an explicit expression for $d(K)$ in terms of m, n and S [2, Corollary 4]. Using the discriminant-conductor formula, it is easy to deduce from their result an explicit expression for the conductor $f(K)$ of K in terms of m, n and S , see (13)–(15). From this formula for $f(K)$ in terms of m, n and S , it is easy to see what arithmetic relations between m, n, S and A, B, C must be proved in order to deduce the form of $f(K)$ given in Theorem 1, see (16) and (17). The remainder of the proof of Theorem 1 requires a lot of technical but straightforward arithmetic results, see (18)–(56).

TABLE (i)/1: Values of α			
α	congruence conditions		
0	$A \equiv 1(4), \quad B \equiv 0(4), \quad C \equiv 1(2)$		
	$A \equiv 1(4), \quad B \equiv 2(4), \quad C \equiv 0(2)$		
	$A \equiv 3(4), \quad B \equiv 0(4), \quad C \equiv 0(2)$		
	$A \equiv 3(4), \quad B \equiv 2(4), \quad C \equiv 1(2)$		
	$A \equiv 0(2), \quad B \equiv 1(2), \quad C \equiv 1(2)$		
	$A \equiv 2(8), \quad B \equiv 0(16), \quad C \equiv 5(8)$		
	$A \equiv 10(16), \quad B \equiv 8(16), \quad C \equiv 5(8)$		
	$A \equiv 6(8), \quad B \equiv 0(64), \quad C \equiv 1(8), \quad b \geq \ell(\text{even}) \geq 6, \quad (A^2 - 4C)/2^\ell \equiv 1(4)$		
	$A \equiv 6(16), \quad B \equiv 32(64), \quad C \equiv 1(8), \quad (A^2 - 4C)/2^\ell \equiv 1(4)$		
	$A \equiv 6(16), \quad B \equiv 0(128), \quad C \equiv 1(8), \quad \ell(\text{even}) = b + 1 \geq 8, \quad (A^2 - 4C)/2^\ell \equiv 3(4)$		
	$A \equiv 6(16), \quad B \equiv 0(128), \quad C \equiv 1(8), \quad \ell(\text{odd}) = b + 2 \geq 9, \quad (A^2 - 4C)/2^\ell \equiv 3(4)$		
	$A \equiv 14(16), \quad B \equiv 32(64), \quad C \equiv 1(8), \quad (A^2 - 4C)/2^\ell \equiv 3(4)$		
	$A \equiv 14(16), \quad B \equiv 0(128), \quad C \equiv 1(8), \quad \ell(\text{odd}) = b + 2 \geq 9, \quad (A^2 - 4C)/2^\ell \equiv 1(4)$		
2	$A \equiv 1(4), \quad B \equiv 0(4), \quad C \equiv 0(2)$		
	$A \equiv 1(4), \quad B \equiv 2(4), \quad C \equiv 1(2)$		
	$A \equiv 3(4), \quad B \equiv 0(4), \quad C \equiv 1(2)$		
	$A \equiv 3(4), \quad B \equiv 2(4), \quad C \equiv 0(2)$		
	$A \equiv 0(8), \quad B \equiv 0(8), \quad C \equiv 4(8)$		
	$A \equiv 2(8), \quad B \equiv 0(16), \quad C \equiv 1(8), \quad \ell \geq 6$		
	$A \equiv 2(16), \quad B \equiv 8(16), \quad C \equiv 5(8)$		
	$A \equiv 4(8), \quad B \equiv 8(16), \quad C \equiv 4(8)$		
	$A \equiv 6(8), \quad B \equiv 0(16), \quad C \equiv 5(8)$		
	$A \equiv 6(8), \quad B \equiv 0(64), \quad C \equiv 1(8), \quad b \geq \ell(\text{even}) \geq 6, \quad (A^2 - 4C)/2^\ell \equiv 3(4)$		
	$A \equiv 6(16), \quad B \equiv 32(64), \quad C \equiv 1(8), \quad (A^2 - 4C)/2^\ell \equiv 3(4)$		
	$A \equiv 6(16), \quad B \equiv 0(128), \quad C \equiv 1(8), \quad \ell(\text{odd}) = b + 2 \geq 9, \quad (A^2 - 4C)/2^\ell \equiv 1(4)$		
	$A \equiv 6(16), \quad B \equiv 0(128), \quad C \equiv 1(8), \quad \ell(\text{even}) = b + 1 \geq 8, \quad (A^2 - 4C)/2^\ell \equiv 1(4)$		
	$A \equiv 14(16), \quad B \equiv 32(64), \quad C \equiv 1(8), \quad (A^2 - 4C)/2^\ell \equiv 1(4)$		
	$A \equiv 14(16), \quad B \equiv 0(128), \quad C \equiv 1(8), \quad \ell(\text{odd}) = b + 2 \geq 9, \quad (A^2 - 4C)/2^\ell \equiv 3(4)$		
3	$A \equiv 0(4), \quad B \equiv 0(4), \quad C \equiv 1(2)$		
	$A \equiv 2(4), \quad B \equiv 0(8), \quad C \equiv 0(4)$		
	$A \equiv 2(8), \quad B \equiv 0(16), \quad C \equiv 1(8), \quad \ell = 5$		
	$A \equiv 6(8), \quad B \equiv 16(32), \quad C \equiv 1(8)$		
	$A \equiv 6(8), \quad B \equiv 0(64), \quad C \equiv 1(8), \quad \ell(\text{even}) = b + 2 \geq 8$		
	$A \equiv 6(16), \quad B \equiv 0(64), \quad C \equiv 1(8), \quad \ell(\text{odd}) = b + 1 \geq 7$		
	$A \equiv 14(16), \quad B \equiv 0(128), \quad C \equiv 1(8), \quad b \geq \ell(\text{odd}) \geq 7$		
	$A \equiv 4(8), \quad B \equiv 0(16), \quad C \equiv 4(8), \quad b = \ell - 1 \geq 5 \text{ or } b \geq \ell$		

TABLE (i)/2: Values of α		
α	examples	
0	$X^4 - 55X^2 - 60X + 145$	$f(K) = 3 \cdot 5$
	$X^4 - 51X^2 - 34X + 68$	$f(K) = 17$
	$X^4 - 65X^2 - 260X - 260$	$f(K) = 5 \cdot 13$
	$X^4 - 17X^2 - 34X - 17$	$f(K) = 17$
	$X^4 - 26X^2 - 39X + 13$	$f(K) = 3 \cdot 13$
	$X^4 - 182X^2 - 624X - 299$	$f(K) = 3 \cdot 13$
	$X^4 - 102X^2 - 136X + 221$	$f(K) = 17$
	$X^4 - 170X^2 - 1088X - 1751$	$f(K) = 17$
	$X^4 - 170X^2 - 544X + 2329$	$f(K) = 17$
	$X^4 - 490X^2 - 1920X + 9145$	$f(K) = 3 \cdot 5$
	$X^4 - 714X^2 - 2176X + 33881$	$f(K) = 17$
	$X^4 - 130X^2 - 480X + 145$	$f(K) = 3 \cdot 5$
	$X^4 - 2210X^2 - 8320X + 946465$	$f(K) = 5 \cdot 13$
2	$X^4 - 119X^2 - 68X + 5848$	$f(K) = 2^2 \cdot 17$
	$X^4 - 15X^2 - 10X + 5$	$f(K) = 2^2 \cdot 5$
	$X^4 - 45X^2 - 20X + 305$	$f(K) = 2^2 \cdot 5$
	$X^4 - 85X^2 - 102X + 34$	$f(K) = 2^2 \cdot 3 \cdot 17$
	$X^4 - 272X + 884$	$f(K) = 2^2 \cdot 17$
	$X^4 - 102X^2 - 544X + 6953$	$f(K) = 2^2 \cdot 17$
	$X^4 - 30X^2 - 40X + 5$	$f(K) = 2^2 \cdot 5$
	$X^4 - 20X^2 - 40X - 20$	$f(K) = 2^2 \cdot 5$
	$X^4 - 50X^2 - 80X + 205$	$f(K) = 2^2 \cdot 5$
	$X^4 + 102X^2 - 1088X + 2873$	$f(K) = 2^2 \cdot 17$
	$X^4 - 90X^2 - 160X + 905$	$f(K) = 2^2 \cdot 5$
	$X^4 - 330X^2 - 640X + 18905$	$f(K) = 2^2 \cdot 5$
	$X^4 - 170X^2 - 640X + 505$	$f(K) = 2^2 \cdot 5$
	$X^4 - 50X^2 - 160X - 95$	$f(K) = 2^2 \cdot 5$
	$X^4 + 1054X^2 - 2176X + 297313$	$f(K) = 2^2 \cdot 17$
3	$X^4 - 20X^2 - 20X - 5$	$f(K) = 2^3 \cdot 5$
	$X^4 - 50X^2 - 40X + 220$	$f(K) = 2^3 \cdot 5$
	$X^4 - 70X^2 - 240X - 95$	$f(K) = 2^3 \cdot 3 \cdot 5$
	$X^4 - 50X^2 - 80X + 145$	$f(K) = 2^3 \cdot 5$
	$X^4 - 490X^2 - 960X + 43705$	$f(K) = 2^3 \cdot 3 \cdot 5$
	$X^4 - 90X^2 - 320X - 55$	$f(K) = 2^3 \cdot 5$
	$X^4 - 1170X^2 - 16640X - 59215$	$f(K) = 2^3 \cdot 5 \cdot 13$
	$\left\{ \begin{array}{l} X^4 - 60X^2 - 160X + 20 \\ X^4 - 180X^2 - 320X + 4820 \end{array} \right\}$	$f(K) = 2^3 \cdot 5$

TABLE (i)/3: Values of α	
α	congruence conditions
4	$A \equiv 0(8), B \equiv 0(8), C \equiv 0(8)$
	$A \equiv 0(8), B \equiv 0(8), C \equiv 2(4)$
	$A \equiv 4(8), B \equiv 0(16), C \equiv 2(8)$
	$A \equiv 4(8), B \equiv 0(16), C \equiv 4(8), b = \ell - 1 = 4 \text{ or } b \leq \ell - 2$

TABLE (i)/4: Values of α	
α	examples
4	$X^4 - 24X^2 - 32X + 8 \quad f(K) = 2^4$
	$X^4 - 8X^2 - 8X - 2 \quad f(K) = 2^4$
	$X^4 - 20X^2 - 16X + 34 \quad f(K) = 2^4$
	$\left\{ \begin{array}{l} X^4 - 12X^2 - 16X - 4 \\ X^4 - 20X^2 - 32X + 4 \end{array} \right\} \quad f(K) = 2^4$

TABLE (ii): Values of β		
β	conditions	examples
0	$v_2(B) = 0$	$X^4 + 10X^2 + 25X + 25 \quad f(K) = 5$
2	$v_2(B) \equiv 1(2)$	$X^4 + 442X^2 - 9248X + 48841 \quad f(K) = 2^2 \cdot 17$
3	$v_2(B) = 4$	$X^4 + 190X^2 + 400X + 9025 \quad f(K) = 2^3 \cdot 5$
4	$v_2(B) = 6$	$X^4 + 28X^2 + 64X + 196 \quad f(K) = 2^4$

TABLE (iii): Values of γ		
γ	congruence conditions	examples
0	$A \equiv 1(4), C \equiv 1(2)$	$X^4 - 15X^2 + 45 \quad f(K) = 3 \cdot 5$
	$A \equiv 3(4), C \equiv 0(4)$	$X^4 - 17X^2 + 68 \quad f(K) = 17$
	$A \equiv 2(8), C \equiv 5(8)$	$X^4 - 78X^2 + 1053 \quad f(K) = 3 \cdot 13$
	$A \equiv 6(8), C \equiv 1(8)$	$X^4 - 34X^2 + 17 \quad f(K) = 17$
2	$A \equiv 1(4), C \equiv 0(4)$	$X^4 - 51X^2 + 612 \quad f(K) = 2^2 \cdot 3 \cdot 17$
	$A \equiv 3(4), C \equiv 1(2)$	$X^4 - 5X^2 + 5 \quad f(K) = 2^2 \cdot 5$
	$A \equiv 2(8), C \equiv 1(8)$	$X^4 + 34X^2 + 17 \quad f(K) = 2^2 \cdot 17$
	$A \equiv 6(8), C \equiv 5(8)$	$X^4 - 10X^2 + 5 \quad f(K) = 2^2 \cdot 5$
3	$A \equiv 2(4), C \equiv 0(4)$	$X^4 - 10X^2 + 20 \quad f(K) = 2^3 \cdot 5$
	$A \equiv 4(8), C \equiv 4(16)$	$X^4 - 68X^2 + 68 \quad f(K) = 2^3 \cdot 17$
4	$A \equiv 4(8), C \equiv 2(8)$	$X^4 - 4X^2 + 2 \quad f(K) = 2^4$
	$A \equiv 8(16), C \equiv 8(32)$	$X^4 - 8X^2 + 8 \quad f(K) = 2^4$

By [3: Theorem 1 (iv)] the cubic resolvent $c(X) = X^3 - AX^2 - 4CX + (4AC - B^2)$ of $q(X)$ has exactly one root $t \in Z$. Thus we have

$$(2) \quad (t - A)(t^2 - 4C) = B^2.$$

Clearly we see that $t - A \neq 0$, $t^2 - 4C \neq 0$, as $B \neq 0$. Solving the quartic equation $\theta^4 + A\theta^2 + B\theta + C = 0$ we find

$$(3) \quad \theta = \frac{\varepsilon(t - A) + \delta \sqrt{(A^2 - t^2) - 2B\varepsilon\sqrt{t - A}}}{2\sqrt{t - A}},$$

where $\varepsilon = \pm 1$, $\delta = \pm 1$. If $t - A \in Z^2$ then we have $[K : Q] = [Q(\theta) : Q] = 1$ or 2, contradicting $[K : Q] = 4$. Hence $t - A \notin Z^2$ and we can write

$$(4) \quad t - A = R^2S,$$

where $S(\neq 1)$ is squarefree. From (2) and (4) we see that $RS \mid B$ so that

$$(5) \quad B = B_1RS,$$

$$(6) \quad t^2 - 4C = B_1^2S.$$

From (4) and (6) we obtain

$$(7) \quad A^2 - 4C = S(B_1^2 - R^2(t + A)).$$

The unique quadratic subfield of K is

$$(8) \quad k = Q(\sqrt{t - A}) = Q(\sqrt{S}).$$

As k is real, we have $S \geq 2$. The splitting field of the cubic resolvent

$$c(X) = (X - t)(X^2 + (t - A)X + (t^2 - At - 4C))$$

is

$$Q\left(\sqrt{(t - A)^2 - 4(t^2 - At - 4C)}\right) = Q\left(\sqrt{-3t^2 + 2At + (A^2 + 16C)}\right).$$

Since K is cyclic, by [3: Theorem 1 (iv)], we must have

$$Q\left(\sqrt{-3t^2 + 2At + (A^2 + 16C)}\right) = k = Q(\sqrt{S}),$$

so there exists an integer z such that

$$(9) \quad -3t^2 + 2At + (A^2 + 16C) = Sz^2.$$

Equivalent forms of (9) are

$$(9)' \quad (t + A)^2 - 4(t^2 - 4C) = Sz^2,$$

$$(9)'' \quad (t - A)^2 - 4t(t - A) + 16C = Sz^2.$$

Further, from (3), we see that

$$\begin{aligned}
K = Q(\theta) &= Q\left(\sqrt{(A^2 - t^2) - 2B\varepsilon\sqrt{t - A}}\right) \\
&= Q\left(\sqrt{(A^2 - t^2) + 2B\sqrt{t - A}}\right) \\
&= Q\left(\sqrt{-R^2S(t + A) + 2B_1R^2S\sqrt{S}}\right), \quad \text{by (4), (5),} \\
&= Q\left(\sqrt{-(t + A) + 2B_1\sqrt{S}}\right).
\end{aligned}$$

Now let M^2 denote the largest square dividing both $t + A$ and $2B_1$. Set

$$(10) \quad t + A = -M^2m, \quad 2B_1 = M^2n,$$

so that

$$(11) \quad (m, n) \text{ is squarefree,}$$

and

$$(12) \quad K = Q\left(\sqrt{m + n\sqrt{S}}\right).$$

Appealing to [2, Corollary 4], as well as the conductor-discriminant formula, we obtain

$$f(K) = 2^\lambda \frac{(m, n)S}{(m, n, S)},$$

where the values of λ are given in TABLE (iv).

Thus

$$(13) \quad f(K) = f_E(K)f_O(K),$$

where the 2-part $f_E(K)$ of $f(K)$ is

$$(14) \quad f_E(K) = \begin{cases} 2^\lambda, & \text{if } 2 \nmid (m, n), \ 2 \nmid S, \\ 2^{\lambda+1}, & \text{otherwise,} \end{cases}$$

and the odd part $f_O(K)$ of $f(K)$ is

$$(15) \quad f_O(K) = \prod_{(p|S) \text{ or } \substack{p \neq 2 \\ (p \nmid S, p|(m, n))}} p,$$

where p runs through primes.

TABLE (iv): Values of λ	
λ	congruence conditions
-1	$m \equiv 2 \pmod{8}, n \equiv 2 \pmod{4}, S \equiv 1 \pmod{8}$ $m \equiv 6 \pmod{8}, n \equiv 2 \pmod{4}, S \equiv 5 \pmod{8}$
0	$m \equiv 1 \pmod{4}, n \equiv 0 \pmod{4}, S \equiv 1 \pmod{8}$ $m \equiv 3 \pmod{4}, n \equiv 2 \pmod{4}, S \equiv 5 \pmod{8}$
1	$m \equiv 6 \pmod{8}, n \equiv 2 \pmod{4}, S \equiv 1 \pmod{8}$ $m \equiv 2 \pmod{8}, n \equiv 2 \pmod{4}, S \equiv 5 \pmod{8}$
2	$m \equiv 2 \pmod{4}, n \equiv 0 \pmod{4}, S \equiv 1 \pmod{4}$ $m \equiv 3 \pmod{4}, n \equiv 0 \pmod{4}, S \equiv 1 \pmod{8}$ $m \equiv 1 \pmod{4}, n \equiv 2 \pmod{4}, S \equiv 5 \pmod{8}$
3	$m \equiv 1 \pmod{2}, n \equiv 1 \pmod{2}, S \equiv 1 \pmod{4}$ $m \equiv 4 \pmod{8}, n \equiv 2 \pmod{4}, S \equiv 2 \pmod{8}$ $m \equiv 2 \pmod{4}, n \equiv 1 \pmod{2}, S \equiv 2 \pmod{8}$

Thus, to complete the proof, we must show that

$$(16) \quad \alpha = \begin{cases} \lambda, & \text{if } 2 \nmid (m, n), 2 \nmid S, \\ \lambda + 1, & \text{otherwise,} \end{cases}$$

where the values of α are given in TABLE (i), and that for odd primes p we have

$$(17) \quad (p \mid S) \text{ or } (p \mid m, p \mid n, p \nmid S) \\ \iff (e_p \equiv 1 \pmod{2}) \text{ or } (e_p \equiv 0 \pmod{2}, e_p \geq 2, p \mid A),$$

where $e_p = \min(v_p(A^2 - 4C), v_p(B))$. We prove (17) first and then (16).

PROOF of (17). Although we use b for $v_2(B)$ and ℓ for $v_2(A^2 - 4C)$, just for the proof of (17), we set for an odd prime p

$$(18) \quad b = v_p(B), \ell = v_p(A^2 - 4C)$$

and

$$(19) \quad b_1 = v_p(B_1), u = v_p(t + A).$$

We need a number of preliminary results ((20) to (45) below). By (5) we have

$$(20) \quad 0 \leq b_1 \leq b$$

and

$$(21) \quad v_p(R) = \begin{cases} b - b_1, & \text{if } p \nmid S, \\ b - b_1 - 1, & \text{if } p \mid S. \end{cases}$$

Further, from (4), we see that

$$(22) \quad v_p(t - A) = \begin{cases} 2(b - b_1), & \text{if } p \nmid S, \\ 2(b - b_1) - 1, & \text{if } p \mid S, \end{cases}$$

and, from (6), that

$$(23) \quad v_p(t^2 - 4C) = \begin{cases} 2b_1, & \text{if } p \nmid S, \\ 2b_1 + 1, & \text{if } p \mid S. \end{cases}$$

Considering the power of p in both sides of (7), we see that exactly one of the following three possibilities must occur

$$(24) \quad \begin{cases} \ell = 2x < 2(b - x) + u, & \text{if } p \nmid S, \\ \ell - 1 = 2x < 2(b - x - 1) + u, & \text{if } p \mid S, \end{cases}$$

$$(25) \quad \begin{cases} 2x > 2(b - b_1) + u = \ell, & \text{if } p \nmid S, \\ 2x > 2(b - b_1 - 1) + u = \ell - 1, & \text{if } p \mid S, \end{cases}$$

$$(26) \quad \begin{cases} 2x = 2(b - b_1) + u \leq \ell, & \text{if } p \nmid S, \\ 2x = 2(b - b_1 - 1) + u \leq \ell - 1, & \text{if } p \mid S. \end{cases}$$

From (24), (25) and (26), we see immediately that

$$(27) \quad (p \nmid S, \ell \equiv 1 \pmod{2}) \text{ or } (p \mid S, \ell \equiv 0 \pmod{2}) \\ \implies (24) \text{ cannot occur}$$

$$(28) \quad (p \nmid S, \ell \not\equiv u \pmod{2}) \text{ or } (p \mid S, \ell \equiv u \pmod{2}) \\ \implies (25) \text{ cannot occur,}$$

$$(29) \quad u \equiv 1 \pmod{2} \implies (26) \text{ cannot occur.}$$

Next, from (10), (11) and (19), we see that

$$(30) \quad u \equiv 1 \pmod{2}, b_1 \geq u \implies p \mid (m, n),$$

$$(31) \quad x \equiv 1 \pmod{2}, b_1 \leq u \implies p \mid (m, n),$$

$$(32) \quad u \equiv 0 \pmod{2}, b_1 \geq u \implies p \nmid m,$$

$$(33) \quad x \equiv 0 \pmod{2}, b_1 \leq u \implies p \nmid n.$$

From (5) and (10) we have

$$(34) \quad p \nmid B \implies p \nmid S, p \nmid n.$$

From (7) and (10) we have

$$(35) \quad \ell = 0 \implies p \nmid S, p \nmid (m, n).$$

From (5) and (7) we have

$$(36) \quad b \geq 1, \ell \geq 1, p \nmid S \implies b_1 \geq 1.$$

From (10) and (20) we have

$$(37) \quad u = 0 \implies p \nmid m.$$

Next we show that

$$(38) \quad p \nmid S, b \geq 1, \ell \geq 1, u = 0 \implies p \nmid A.$$

Suppose $p \mid A$. Then, by (18), we have $p \mid B$, $p \mid A^2 - 4C$, $p \mid C$. As $p \nmid S$, by (5), p divides one of B_1 and R . By (7) p must divide both of B_1 and R . Hence, by (4), we have $p \mid t - A$ and thus, by (9)'', $p \mid z$. By (6) we have $p \mid t^2 - 4C$ and so, by (9)', $p \mid t + A$, contradicting $u = 0$. This completes the proof of (38).

Our next result asserts that

$$(39) \quad p \nmid A, u \geq 1 \implies b_1 = b.$$

As $p \nmid A$ and $u \geq 1$ we have $p \nmid t - A$, so that, by (4), we have $p \nmid RS$, and thus, by (5), $b_1 = b$. This completes the proof of (39).

We now prove that

$$(40) \quad p \nmid S, p \nmid A, \ell \geq 2 \implies u \neq 1.$$

Suppose $u = 1$, that is, $p \parallel t + A$. By (7) we see that $p \mid B_1$ and $p \mid R$. Then, by (4), we have $p \mid t - A$ and so $p \mid A$, contradicting $p \nmid A$. This completes the proof of (40).

We next show that

$$(41) \quad p \nmid S, b_1 \geq 2, u \geq 2 \implies b_1 = b.$$

Suppose $b_1 \neq b$. By (20) and (21) we have $p \mid R$. Then, by (4), we have $p^2 \mid t - A$, so that as $p^2 \mid t + A$ we have $p^2 \mid t$ and $p^2 \mid A$. Further, as

$p^2 \mid B_1$, $p \mid R$, from (5), we see that $p^3 \mid B$. Then, from (6), as $p^4 \mid t^2$ and $p^4 \mid B_1^2$, we see that $p^4 \mid C$. This contradicts (1) and so we must have $b_1 = b$ as claimed.

Next we prove that

$$(42) \quad p \nmid A \implies p \nmid S.$$

Suppose $p \nmid A$ yet $p \mid S$. Then, by (4), we have $p \mid t - A$, and, by (6), we deduce $p \mid t^2 - 4C$. Then, appealing to (9)', we see that $p \mid t + A$. Hence we have $p \mid A$, which is a contradiction, proving (42).

We now show that

$$(43) \quad p \nmid S, u = 1 \implies \ell \leq b.$$

We know that exactly one of the possibilities (24), (25), (26) must occur. If (24) holds with $u = 1$ then $\ell = 2b_1 < 2(b - b_1) + 1$, so $\ell = 2b_1 \leq 2(b - b_1)$, that is, $\ell = 2b_1 \leq b$. If (25) holds with $u = 1$ then $\ell = 1 + 2(b - b_1) < 2b_1$, so $\ell = 1 + 2(b - b_1) \leq 2b_1 - 1$, and thus $\ell = 1 + 2b - 2b_1 \leq b$. The possibility (26) cannot occur with $u = 1$ by (29). This completes the proof of (43).

Next we prove

$$(44) \quad p \nmid S, u = 0 \implies \begin{cases} \ell < b, & \text{if (24) or (25) holds,} \\ \ell \geq b, & \text{if (26) holds.} \end{cases}$$

If (24) holds with $u = 0$ then $2b_1 < 2(b - b_1)$, $2b_1 < b$, $\ell < b$. If (25) holds with $u = 0$ then $2b_1 > 2(b - b_1)$, $2b_1 > b$, $\ell = 2(b - b_1) < b$. If (26) holds with $u = 0$ then $2b_1 = 2(b - b_1)$, $b = 2b_1 \leq \ell$. This completes the proof of (44).

Our last preliminary result is the following

$$(45) \quad p \nmid S, \quad b = b_1, \quad u \geq 1 \implies p \nmid A.$$

As $b = b_1$, by (21), we have $p \nmid R$. Hence, by (4), we deduce $p \nmid t - A$. But $u \geq 1$ so that $p \mid t + A$. Thus we must have $p \nmid A$ as asserted.

We are now ready to prove (17). We do this by justifying the assertions of TABLE (v) above.

Cases 1 and 2 of TABLE (v) follow immediately from (34) and (35). It remains to treat cases 3–18. For these cases we have $b \geq 1$ and $\ell \geq 1$. To complete the proof of the table we must show that

$$(46) \quad p \nmid S, \text{ cases } 3, 5, 6, 7, 9, 10, 11 \text{ } (v_p(C) \text{ even}),$$

$$13, 14, 15 \text{ } (v_p(C) \text{ even}), 17, 18,$$

$$(47) \quad p \mid S, \text{ cases } 4, 8, 11 \text{ } (v_p(C) \text{ odd}), 12, 15 \text{ } (v_p(C) \text{ odd}), 16,$$

$$(48) \quad \begin{cases} p \mid (m, n), & \text{cases } 3, 7, 10, 11 \text{ } (v_p(C) \text{ even}), \\ & 13, 15 \text{ } (v_p(C) \text{ even}), 17, 18, \\ p \nmid (m, n), & \text{cases } 5, 6, 9, 14. \end{cases}$$

Clearly (46) follows from (42) in cases 5, 6, 9, 10, 13, 14, 17, 18. We establish (46) for cases 3 and 7 by proving that

$$b \geq \ell(\text{even}) \geq 2, \quad p \mid A \implies p \nmid S.$$

We assume that $p \parallel S$ and obtain a contradiction. As $p \mid S$, by (4), we see that $p \mid t - A$, and thus $p \mid t + A$. If $p \parallel t - A$ then by (4) $p \nmid R$. Hence by (5) $p^{b-1} \parallel B_1$ so that by (6) $p^{2b-1} \parallel t^2 - 4C$. As $b \geq \ell > 1$ we have $2b - 1 > \ell$ so that $p^\ell \mid p^{2b-1} \parallel SB_1^2$. Hence by (7) we see that $p^\ell \parallel SR^2(t + A)$, that is, $p^{\ell-1} \parallel t + A$. It is clear from (9)' that $v_p((t + A)^2 - 4(t^2 - 4C)) = v_p(Sz^2) \equiv 1 \pmod{2}$ so that

$$\min(2(\ell - 1), 2b - 1) = 2b - 1,$$

implying $b \leq \ell - 1$, which contradicts $b \geq \ell$. If $p \parallel t + A$ then as $p \mid A$ we have $p \mid t$. Next, as $\ell \geq 2$, we have $p^2 \mid A^2 - 4C$ so $p^2 \mid C$, and thus $p^2 \mid t^2 - 4C$. By (6), $v_p(t^2 - 4C) = v_p(B_1^2 S) \equiv 1 \pmod{2}$ so that $p^3 \mid t^2 - 4C$. Then, by (9)', we see that $v_p((t + A)^2 - 4(t^2 - 4C)) = 2$, contradicting that $v_p(Sz^2) \equiv 1 \pmod{2}$. Hence we must have $p^2 \mid t - A$ and $p^2 \mid t + A$. Thus $p^2 \mid A$ and, by (4), we have $p \mid R$. Next, as $\ell \geq 2$, from (7) we see that $p \mid B_1$, and thus, by (5), $p^3 \mid B$. Then, from (7), we see that $p^3 \mid A^2 - 4C$. But ℓ is even so $p^4 \mid A^2 - 4C$ and thus $p^4 \mid C$, contradicting (1).

We establish (46) for cases 11 and 15 when $v_p(C)$ is even by proving that

$$b \geq \ell(\text{odd}) \geq 1, \quad p \mid A, \quad p^{2k} \parallel C \implies p \nmid S.$$

As $\ell \geq 1$ we have $p \mid A^2 - 4C$ so that $p \mid C$, and thus $k \geq 1$. Hence $p^2 \mid C$ so $p^2 \mid A^2 - 4C$ showing that $\ell \geq 2$. But ℓ is odd so we must have $\ell \geq 3$. Further, as $p^\ell \parallel A^2 - 4C$, where ℓ is odd, and $p^{2k} \parallel C$, we see that $p^{2k} \parallel A^2$, that is $p^k \parallel A$. Moreover, as $b \geq \ell \geq 3$, we have $p^3 \mid B$. If $k \geq 2$ then $p^2 \mid A$, $p^3 \mid B$, $p^4 \mid C$, contradicting (1). Hence we must have $k = 1$, that is $p \parallel A$ and $p^2 \parallel C$. Suppose now that $p \mid S$, so that $p \parallel S$, we will obtain a contradiction. We consider two cases according as $p \nmid R$ or $p \mid R$. If $p \nmid R$ then by (4) we have $p \parallel t - A$. From (5) we see that $p^{b-1} \parallel B_1$, so that $p^{2b-1} \mid SB_1^2$, where $2b-1 \geq 2\ell-1 > \ell$. Hence from (7) we deduce that $p^\ell \parallel SR^2(t+A)$, that is, $p^{\ell-1} \parallel t+A$. From (6) we see that $p^{2b-1} \parallel t^2 - 4C$. Then, from (9)', as Sz^2 is divisible by an odd power of p , we deduce that $2b-1 < 2\ell-2$, that is, $b \leq \ell-1$, which contradicts $b \geq \ell$. We now turn to the case $p \mid R$, say, $p^r \parallel R$, where $r \geq 1$. From (4) we deduce that $p^{2r+1} \parallel t - A$. As $p \parallel A$ and $p^3 \mid t - A$ we have $p \parallel t + A$. From (5) we deduce that $p^{b-r-1} \parallel B_1$, so that by (6) $p^{2(b-r-1)+1} \parallel t^2 - 4C$. Then, from (9)', as Sz^2 is divisible by an odd power of p , we must have $2(b-r-1)+1 = 1$, that is $r = b-1$, and hence $p \parallel t^2 - 4C$. On the other hand we have $p \mid t$ and $p^2 \mid C$ so that $p^2 \mid t^2 - 4C$, which is the required contradiction. This completes the proof of (46).

Next we prove (47). First we treat cases 4 and 12. We prove

$$(49)_1 \quad b(\text{even}) \geq 2, \quad b < \ell, \quad p \mid A, \quad p^i \parallel C \quad (i = 2, 3) \implies p \mid S$$

and

$$(49)_2 \quad b(\text{even}) \geq 2, \quad b < \ell, \quad p \mid A,$$

$$p^i \parallel C \quad (i = 0, 1 \text{ or } i \geq 4) \text{ cannot occur.}$$

$i = 0, 1$. Here $\ell > b \geq 2$ so $p^2 \mid A^2 - 4C$. But $p \mid A$, so $p^2 \mid A^2$, and thus $p^2 \mid C$, contradicting $i = 0, 1$. This case cannot occur.

$i = 2$. Here $p^2 \parallel C$, $\ell > b \geq 2$ so $\ell \geq 3$, $p^3 \mid A^2 - 4C$, and thus $p \parallel A$. Assume $p \nmid S$. Then, by (5), we have $p \mid B_1$ or $p \mid R$. If $p \nmid R$, so that $p \mid B_1$,

we have by (4) $p \nmid t - A$. But, by (7), we have $p^2 \mid t + A$, contradicting $p \mid A$. Hence we must have $p \mid R$. Then, by (7), we see that $p \mid B_1$. By (4) we have $p^2 \mid t - A$ so, as $p \parallel A$, we have $p \parallel t + A$, that is $u = 1$. Hence, by (43), we have $\ell \leq b$, contradicting $b < \ell$. Thus we must have $p \mid S$ in this case.

$i = 3$. Here $p^3 \parallel C$, $\ell > b \geq 2$, $\ell \geq 3$, $p^3 \mid A^2 - 4C$, so that $p^2 \mid A$. Assume $p \nmid S$. Then, by (5), we have $p \mid B_1$ or $p \mid R$. If $p \nmid R$, so that $p \mid B_1$, by (4) we have $p \nmid t - A$. But, by (7), we have $p^2 \mid t + A$ contradicting $p \mid A$. Hence we must have $p \mid R$. Then, by (7), we see that $p \mid B_1$. From (6), we see that $p^3 \parallel t^2 - SB_1^2$, so that $p \parallel B_1$, $p \parallel t$. Hence we have $p^2 \parallel S(B_1^2 - R^2(t + A))$, contradicting $p^3 \mid A^2 - 4C$. Thus we must have $p \mid S$ in this case.

$i \geq 4$. As $\ell > b \geq 2$, we have $\ell \geq 3$, so $p^3 \mid A^2 - 4C$. But $p^4 \mid C$, so $p^3 \mid A^2$, $p^2 \mid A$. Now $p^2 \mid B$ so, by (5), we have either $p \mid R$ or $p \nmid R$, $p \mid B_1$. Suppose $p \mid R$. Then, by (4), we have $p^2 \mid t - A$, and thus $p^2 \mid t + A$, $p^4 \mid R^2(t + A)$, so that $p^3 \mid SB_1^2$ by (7). If $p \mid S$ then $p \mid B_1$, $p^3 \mid B$, contradicting (1). If $p \nmid S$ then $p^3 \mid B_1^2$, $p^2 \mid B_1$, $p^3 \mid B$, contradicting (1). Thus we must have $p \nmid R$, $p \mid B_1$. By (7) we have $p^2 \mid t + A$, so $p^2 \mid t - A$, $p^2 \mid R^2S$, $p \mid R$, contradicting $p \nmid R$. Thus this case cannot occur. This completes the proof of (49), and hence of (47), for cases 4 and 12.

We now prove (47) for cases 8 and 16. We prove

$$\ell > b(\text{odd}) \geq 1, \quad p \mid A \implies p \mid S.$$

Assume that $p \nmid S$. As $\ell \geq 2$ we have $p^2 \mid A^2 - 4C$ so that $p^2 \mid C$. As $b \geq 1$ we have $p \mid B$ so by (2) either $p \mid t - A$ or $p \mid t^2 - 4C$. For both possibilities we must have $p \mid t$, so that $p \mid t - A$, $p \mid t + A$, $p^2 \mid t^2 - 4C$. Hence $u = v_p(t + A) \geq 1$. If $u = 1$, by (43), we have $\ell \leq b$ contradicting $\ell > b$. Hence $u \geq 2$ so that $p^2 \mid t + A$. From (6) we deduce $p \mid B_1$, and from (4) that $p \mid R$ and $p^2 \mid t - A$. Hence $p^2 \mid A$. From (5) we see that $p^2 \mid B$ so that $b \geq 2$. But b is odd so $b \geq 3$, and $p^3 \mid B$. As $\ell > b \geq 3$ we have $\ell \geq 4$ so $p^4 \mid A^2 - 4C$, and thus $p^4 \mid C$, contradicting (1). This completes the proof of (47) for cases 8 and 16.

We now prove (47) for cases 11 and 15 when $v_p(C)$ is odd by proving that

$$b \geq \ell(\text{odd}) \geq 1, \quad p \mid A, \quad p^{2k+1} \parallel C \implies p \mid S.$$

Let $a = v_p(A)$ so that $p^a \parallel A$, where $a \geq 1$. As $p^\ell \parallel A^2 - 4C$, where ℓ is odd, $p^{2a} \parallel A^2$ and $p^{2k+1} \parallel C$, we must have $\ell = 2k+1 < 2a$. If $k \geq 2$ then $b \geq \ell \geq 5$ and $a \geq 3$, so that $p^3 \mid A$, $p^5 \mid B$, $p^5 \mid C$, which contradicts (1). Hence we must have $k = 0$ or $k = 1$ that is $\ell = 1$ or $\ell = 3$. We suppose that $p \nmid S$ and obtain a contradiction. We consider two cases according as $p \nmid R$ or $p \mid R$. If $p \nmid R$ then by (4) we see that $p \nmid t - A$. As $p \mid A$ we have $p \nmid t$. On the other hand as $p \mid B$ and $p \nmid t - A$ from (2) we see that $p \mid t^2 - 4C$, so that as $p \mid C$, we have the contradiction $p \mid t$. If $p \mid R$ then $p^r \parallel R$ for some $r \geq 1$. From (4) we deduce that $p^{2r} \parallel t - A$ and thus as $p \mid A$ we have $p \mid t$ and $p \mid t + A$. From (5) we obtain $p^{b-r} \parallel B_1$. Thus, from (7), as

$$\begin{aligned} p^\ell \parallel A^2 - 4C \quad (\ell = 1 \text{ or } 3), \quad p^{2(b-r)} \parallel SB_1^2, \\ p^{2r+v_p(t+A)} \mid SR^2(t+A), 2r+v_p(t+A) \geq 3, \end{aligned}$$

we must have

$$\ell = 3, \quad b - r \geq 2, \quad 2r + v_p(t + A) = 3.$$

Hence

$$k = 1, \quad a \geq 2, \quad r = v_p(t + A) = 1, \quad b \geq 3,$$

and thus

$$\begin{aligned} p^3 \parallel C, \quad p \parallel R, \quad p^2 \parallel t - A, \quad p \parallel t + A, \\ p^2 \mid A, \quad p \parallel t, \quad p^2 \parallel t^2 - 4C, \quad p \parallel B_1 \quad (\text{by (6)}), \end{aligned}$$

$p^2 \parallel B$ (by (5)), $b = 2$, contradicting $b \geq 3$. This completes the proof of (47).

We now prove (48). Let p be an odd prime with $p \nmid S$, so that we are in cases 3, 5–7, 9–10, 11 ($v_p(C)$ even), 13–14, 15 ($v_p(C)$ even), 17–18. By (36) we have $x \geq 1$. Exactly one of (24), (25), (26) occurs.

We begin by supposing that (24) occurs, so ℓ is even, and we are in cases 3, 5–7, 9–10. (48) follows from the table below.

	cases	assertion	reason
$u = 0$	3,7,	cannot occur	(38)
	6, 10	cannot occur	(44)
	5,9	$p \nmid m$	(32)
$u = 1$	3,7	$p \mid (m, n)$	(30)
	6, 10	cannot occur	(43)
	5,9	cannot occur	(40)
$u \geq 2, b_1 = 1$	3,7, 10	$p \mid (m, n)$	(31)
	6	cannot occur	(24)
	5,9	cannot occur	(39)
$u \geq 2, b_1 \geq 2$	3,5,7,9	cannot occur	$\ell = 2b_1 = 2b > b(24), (41)$
	10	$p \mid (m, n)$	(24), (31), (41)
	6	$p \nmid n$	(24), (33), (41)

Next we suppose that (25) occurs, so that $\ell \equiv u \pmod{2}$. In cases 3, 5–7, 9–10, ℓ and u are both even, whereas, in cases 11, 13–15, 17–18, ℓ and u are both odd. (48) follows from the table below.

	cases	assertion	reason
$u = 0$	3, 7,	cannot occur	(38)
	11, 13, 14, 15, 17, 18	cannot occur	u odd
	6, 10	cannot occur	(44)
	5, 9	$p \nmid m$	(32)
$u = 1$	11, 13, 15, 17, 18	$p \mid (m, n)$	(30)
	14	cannot occur	(43)
	3, 5, 6, 7, 9, 10	cannot occur	u even
$u \geq 2, b_1 = 1$	3, 7, 10, 11, 13, 15, 17, 18	$p \mid (m, n)$	(31)
	5, 6, 9, 14	cannot occur	(39)
$u \geq 2, b_1 \geq 2$	10, 18	$p \mid (m, n)$	(25), (31), (41)
	6, 14	$p \nmid n$	(25), (33), (41)
	5, 9	$p \nmid m$	(25), (32), (41)
	11, 13, 15, 17	$p \mid (m, n)$	(25), (30), (41)
	3, 7	cannot occur	(41), (45)

Finally we suppose that (26) occurs, so that u is even. (48) follows from the table below.

	cases	assertion	reason
$u = 0$	5, 6, 14	$p \nmid m$	(37)
	7, 9, 11, 13	cannot occur	(44)
	3, 15	cannot occur	(38)
	10, 17, 18	cannot occur	(26)
$u \geq 2, b_1 = 1$	3, 7, 10, 11, 13, 15, 17, 18	$p \mid (m, n)$	(31)
	5, 6, 9, 14	cannot occur	(39)
$u \geq 2, b_1 \geq 2$	3, 5, 7, 9, 11, 13, 15, 17	cannot occur	(26), (41)
	6, 14	$p \nmid n$	(26), (33), (41)
	10, 18	$p \mid (m, n)$	(26), (31), (41)

This completes the proof of (17).

PROOF of (16). We treat each of the cases specified in TABLE (iv) separately. We just give the details for the case

$$m \equiv 2 \pmod{8}, \quad n \equiv 2 \pmod{4}, \quad S \equiv 1 \pmod{8},$$

as this serves as a model for the rest of the cases. Recall that $2^b \parallel B$, $2^\ell \parallel A^2 - 4C$. We define the integers r and μ by $2^r \parallel R$, $2^\mu \parallel M$, so that

$$(50) \quad \left\{ \begin{array}{ll} R \equiv 2^r \pmod{2^{r+1}}, \\ R^2 \equiv 2^{2r} \pmod{2^{2r+3}}, \\ t - A \equiv 2^{2r} \pmod{2^{2r+3}}, & \text{by (4),} \\ M \equiv 2^\mu \pmod{2^{\mu+1}}, \\ t + A \equiv -2^{2\mu+1} \pmod{2^{2\mu+3}}, & \text{by (10),} \\ B_1 \equiv 2^{2\mu} \pmod{2^{2\mu+1}}, & \text{by (10),} \\ b = 2\mu + r, & \text{by (5).} \end{array} \right.$$

From the congruences for $t - A$ and $t + A$, we obtain the following congru-

ences:

$$(51) \quad \begin{cases} t \equiv -2^{2\mu} \pmod{2^{2\mu+2}}, \\ A \equiv -2^{2\mu} \pmod{2^{2\mu+2}}, & \text{if } r \geq \mu + 2, \\ t \equiv 2^{2\mu} \pmod{2^{2\mu+2}}, \\ A \equiv 2^{2\mu} \pmod{2^{2\mu+1}}, & \text{if } r = \mu + 1, \\ t \equiv -2^{2\mu-1} \pmod{2^{2\mu+2}}, \\ A \equiv 5 \cdot 2^{2\mu-1} \pmod{2^{2\mu+1}}, & \text{if } r = \mu, \\ t \equiv 2^{2r-1} \pmod{2^{2r+2}}, \\ A \equiv -2^{2r-1} \pmod{2^{2r+2}}, & \text{if } r \leq \mu - 1. \end{cases}$$

Appealing to (7) we see that there are integers g and h such that

$$A^2 - 4C = (8g + 1)2^{4\mu} + (4h + 1)2^{2r+2\mu+1},$$

so that

$$(52) \quad \ell = \begin{cases} 4\mu, & \text{if } r \geq \mu, \\ 2r + 2\mu + 1, & \text{if } r \leq \mu - 1, \end{cases}$$

and

$$(53) \quad (A^2 - 4C)/2^\ell \equiv \begin{cases} 1 \pmod{8}, & \text{if } r \geq \mu + 1, \\ 3 \pmod{8}, & \text{if } r = \mu, \\ 3 \pmod{4}, & \text{if } r = \mu - 1, \\ 1 \pmod{4}, & \text{if } r \leq \mu - 2. \end{cases}$$

Next, from (6), we obtain

$$(54) \quad \begin{cases} C \equiv 0 \pmod{2^{4\mu+1}}, & \text{if } r \geq \mu + 1, \\ C \equiv 2^{4\mu-4} - 2^{4\mu-2} \pmod{2^{4\mu-1}}, & \text{if } r = \mu, \\ C \equiv 2^{4r-4} \pmod{2^{4r-1}}, & \text{if } r \leq \mu - 1. \end{cases}$$

Thus we have

$$(55) \quad \begin{cases} 2^{2\mu} \parallel A, \ 2^{3\mu+2} \mid B, \ 2^{4\mu+1} \mid C, & \text{if } r \geq \mu + 2, \\ 2^{2\mu} \parallel A, \ 2^{3\mu+1} \mid B, \ 2^{4\mu+1} \mid C, & \text{if } r = \mu + 1, \\ 2^{2\mu-1} \parallel A, \ 2^{3\mu} \parallel B, \ 2^{4\mu-4} \parallel C, & \text{if } r = \mu, \\ 2^{2r-1} \parallel A, \ 2^{3r+2} \mid B, \ 2^{4r-4} \parallel C, & \text{if } r \leq \mu - 1, \end{cases}$$

and so, by (1), we have

$$(56) \quad \begin{cases} \mu = 0, & \text{if } r \geq \mu + 2, \\ \mu = 0, & \text{if } r = \mu + 1, \\ \mu = 1, & \text{if } r = \mu, \\ r = 1, & \text{if } r \leq \mu - 1. \end{cases}$$

Appealing to (50), (51), (52), (53), (54), and (56), we have:

I: $m \equiv 2 \pmod{8}$, $n \equiv 2 \pmod{4}$, $S \equiv 1 \pmod{8}$		
$A \equiv 3 \pmod{4}$,	$B \equiv 0 \pmod{4}$,	$C \equiv 0 \pmod{2}$,
$b \geq 2, \ell = 0$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{8}$,	
$A \equiv 1 \pmod{4}$,	$B \equiv 2 \pmod{4}$,	$C \equiv 0 \pmod{2}$,
$b = 1, \ell = 0$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{8}$,	
$A \equiv 10 \pmod{16}$,	$B \equiv 8 \pmod{16}$,	$C \equiv 5 \pmod{8}$
$b = 3, \ell = 4$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{8}$,	
$A \equiv 14 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$
$b = 5, \ell = 7$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$,	
$A \equiv 14 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{odd}) = b + 2 \geq 9$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$.	

Similarly for the remaining eleven cases in TABLE (iv) we obtain:

II: $m \equiv 6 \pmod{8}$, $n \equiv 2 \pmod{4}$, $S \equiv 5 \pmod{8}$		
$A \equiv 1 \pmod{4}$,	$B \equiv 0 \pmod{4}$,	$C \equiv 1 \pmod{2}$,
$\ell = 0, b \geq 2$,	$(A^2 - 4C)/2^\ell \equiv 5 \pmod{8}$	
$A \equiv 3 \pmod{4}$,	$B \equiv 2 \pmod{4}$,	$C \equiv 1 \pmod{2}$,
$\ell = 0, b = 1$,	$(A^2 - 4C)/2^\ell \equiv 5 \pmod{8}$	
$A \equiv 6 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$\ell = 7, b = 5$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{odd}) = b + 2 \geq 9$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$	
$A \equiv 10 \pmod{16}$,	$B \equiv 8 \pmod{16}$,	$C \equiv 5 \pmod{8}$,
$\ell = 4, b = 3$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{8}$	

III: $m \equiv 1 \pmod{4}$, $n \equiv 0 \pmod{4}$, $S \equiv 1 \pmod{8}$		
$A \equiv 1 \pmod{4}$,	$B \equiv 0 \pmod{4}$,	$C \equiv 1 \pmod{2}$,
$\ell = 0, b \geq 2$,	$(A^2 - 4C)/2^\ell \equiv 5 \pmod{8}$	
$A \equiv 1 \pmod{4}$,	$B \equiv 2 \pmod{4}$,	$C \equiv 0 \pmod{4}$,
$\ell = 0, b = 1$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 3 \pmod{4}$,	$B \equiv 0 \pmod{4}$,	$C \equiv 0 \pmod{2}$,
$\ell = 0, b \geq 2$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{8}$	
$A \equiv 3 \pmod{4}$,	$B \equiv 2 \pmod{4}$,	$C \equiv 3 \pmod{4}$,
$\ell = 0, b = 1$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 6 \pmod{8}$,	$B \equiv 0 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$b \geq \ell(\text{even}) \geq 6$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 6 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$\ell = 7, b = 5$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$b \geq \ell = 6$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{odd}) = b + 2 \geq 9$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$	
$A \equiv 14 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$\ell = 7, b = 5$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$	
$A \equiv 14 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{odd}) = b + 2 \geq 9$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 14 \pmod{16}$,	$B \equiv 0 \pmod{256}$,	$C \equiv 1 \pmod{8}$,
$b \geq \ell(\text{even}) \geq 8$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	

IV: $m \equiv 3 \pmod{4}$, $n \equiv 2 \pmod{4}$, $S \equiv 5 \pmod{8}$		
$A \equiv 0 \pmod{2}$,	$B \equiv 1 \pmod{2}$,	$C \equiv 1 \pmod{2}$,
$\ell \geq 2, b = 0$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$	
$A \equiv 2 \pmod{8}$,	$B \equiv 0 \pmod{16}$,	$C \equiv 5 \pmod{8}$,
$b \geq \ell = 4$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$	
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{even}) = b + 1 \geq 8$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$	
$A \equiv 14 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$\ell = 6, b = 5$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$	

V: $m \equiv 6 \pmod{8}$, $n \equiv 2 \pmod{4}$, $S \equiv 1 \pmod{8}$		
$A \equiv 1 \pmod{4}$,	$B \equiv 0 \pmod{4}$,	$C \equiv 0 \pmod{2}$,
$\ell = 0, b \geq 2, (A^2 - 4C)/2^\ell \equiv 1 \pmod{8}$		
$A \equiv 3 \pmod{4}$,	$B \equiv 2 \pmod{4}$,	$C \equiv 0 \pmod{2}$,
$\ell = 0, b = 1, (A^2 - 4C)/2^\ell \equiv 1 \pmod{8}$		
$A \equiv 2 \pmod{16}$,	$B \equiv 8 \pmod{16}$,	$C \equiv 5 \pmod{8}$,
$\ell = 4, b = 3, (A^2 - 4C)/2^\ell \equiv 7 \pmod{8}$		
$A \equiv 14 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$		
$A \equiv 14 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$		
VI: $m \equiv 2 \pmod{8}$, $n \equiv 2 \pmod{4}$, $S \equiv 5 \pmod{8}$		
$A \equiv 1 \pmod{4}$,	$B \equiv 2 \pmod{4}$,	$C \equiv 1 \pmod{2}$,
$\ell = 0, b = 1, (A^2 - 4C)/2^\ell \equiv 5 \pmod{8}$		
$A \equiv 3 \pmod{4}$,	$B \equiv 0 \pmod{4}$,	$C \equiv 1 \pmod{2}$,
$\ell = 0, b \geq 2, (A^2 - 4C)/2^\ell \equiv 5 \pmod{8}$		
$A \equiv 2 \pmod{16}$,	$B \equiv 8 \pmod{16}$,	$C \equiv 5 \pmod{8}$,
$\ell = 4, b = 3, (A^2 - 4C)/2^\ell \equiv 7 \pmod{8}$		
$A \equiv 6 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$		
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{odd}) = b + 2 \geq 9, (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$		
VII: $m \equiv 2 \pmod{4}$, $n \equiv 0 \pmod{4}$, $S \equiv 1 \pmod{4}$		
$A \equiv 2 \pmod{8}$,	$B \equiv 0 \pmod{16}$,	$C \equiv 1 \pmod{8}$,
$\ell = 5, b \geq 4, (A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$		
$A \equiv 4 \pmod{8}$,	$B \equiv 0 \pmod{32}$,	$C \equiv 4 \pmod{16}$,
$b + 1 \geq \ell \geq 6, (A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$		
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
$\ell(\text{odd}) = b + 1 \geq 7, (A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$		
$A \equiv 14 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
$b \geq \ell(\text{odd}) \geq 7, (A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$		

VIII: $m \equiv 3 \pmod{4}$, $n \equiv 0 \pmod{4}$, $S \equiv 1 \pmod{8}$		
$A \equiv 0 \pmod{8}$,	$B \equiv 0 \pmod{16}$,	$C \equiv 4 \pmod{16}$,
	$b \geq \ell = 4$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$
$A \equiv 2 \pmod{8}$,	$B \equiv 0 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
	$b \geq \ell(\text{even}) \geq 6$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$
$A \equiv 2 \pmod{8}$,	$B \equiv 0 \pmod{32}$,	$C \equiv 1 \pmod{8}$,
	$\ell \geq b(\text{odd}) + 3 \geq 8$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
	$b \geq \ell = 6$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$
$A \equiv 14 \pmod{16}$,	$B \equiv 0 \pmod{256}$,	$C \equiv 1 \pmod{8}$,
	$b \geq \ell(\text{even}) \geq 8$,	$(A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$

IX: $m \equiv 1 \pmod{4}$, $n \equiv 2 \pmod{4}$, $S \equiv 5 \pmod{8}$		
$A \equiv 4 \pmod{8}$,	$B \equiv 8 \pmod{16}$,	$C \equiv 12 \pmod{16}$,
	$\ell = 5, b = 3$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$
$A \equiv 6 \pmod{8}$,	$B \equiv 0 \pmod{16}$,	$C \equiv 5 \pmod{8}$,
	$\ell = 4, b \geq 4$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$
$A \equiv 6 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 1 \pmod{8}$,
	$\ell(\text{even}) = b + 1 \geq 8$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$
$A \equiv 14 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
	$\ell = 6, b = 5$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$

X: $m \equiv 1 \pmod{2}$, $n \equiv 1 \pmod{2}$, $S \equiv 1 \pmod{4}$		
$A \equiv 0 \pmod{4}$,	$B \equiv 4 \pmod{8}$,	$C \equiv 3 \pmod{4}$,
	$\ell = b = 2$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$
$A \equiv 2 \pmod{4}$,	$B \equiv 0 \pmod{8}$,	$C \equiv 0 \pmod{4}$,
	$\ell = 2, b \geq 3$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$
$A \equiv 6 \pmod{8}$,	$B \equiv 16 \pmod{32}$,	$C \equiv 1 \pmod{8}$,
	$\ell \geq 7, b = 4$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$
$A \equiv 6 \pmod{8}$,	$B \equiv 0 \pmod{64}$,	$C \equiv 1 \pmod{8}$,
	$\ell(\text{even}) = b + 2 \geq 8$,	$(A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$

XI: $m \equiv 4 \pmod{8}$, $n \equiv 2 \pmod{4}$, $S \equiv 2 \pmod{8}$		
$A \equiv 4 \pmod{16}$,	$B \equiv 16 \pmod{32}$,	$C \equiv 28 \pmod{32}$,
$\ell = 5, b = 4, (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$		
$A \equiv 8 \pmod{16}$,	$B \equiv 0 \pmod{32}$,	$C \equiv 8 \pmod{32}$,
$\ell = 5, b \geq 5, (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$		
$A \equiv 12 \pmod{16}$,	$B \equiv 64 \pmod{128}$,	$C \equiv 4 \pmod{32}$,
$\ell \geq 10, b = 6, (A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$		
$A \equiv 12 \pmod{16}$,	$B \equiv 0 \pmod{256}$,	$C \equiv 4 \pmod{32}$,
$\ell(\text{odd}) = b + 3 \geq 11, (A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$		
XII: $m \equiv 2 \pmod{4}$, $n \equiv 1 \pmod{2}$, $S \equiv 2 \pmod{8}$		
$A \equiv 0 \pmod{8}$,	$B \equiv 8 \pmod{16}$,	$C \equiv 6 \pmod{8}$,
$\ell = b = 3, (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$		
$A \equiv 4 \pmod{8}$,	$B \equiv 0 \pmod{16}$,	$C \equiv 2 \pmod{8}$,
$\ell = 3, b \geq 4, (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}$		
$A \equiv 12 \pmod{16}$,	$B \equiv 32 \pmod{64}$,	$C \equiv 4 \pmod{32}$,
$\ell = 7, b = 5, (A^2 - 4C)/2^\ell \equiv 3 \pmod{4}$		
$A \equiv 12 \pmod{16}$,	$B \equiv 0 \pmod{128}$,	$C \equiv 4 \pmod{32}$,
$\ell(\text{even}) = b + 3 \geq 10, (A^2 - 4C)/2^\ell \equiv 1 \pmod{2}$		

From these tables, and TABLES (i) and (iv), we obtain the following values of λ and α

I	$\lambda = -1, \quad \alpha = 0$	VII	$\lambda = 2, \quad \alpha = 3$
II	$\lambda = -1, \quad \alpha = 0$	VIII	$\lambda = 2, \quad \alpha = 2$
III	$\lambda = 0, \quad \alpha = 0$	IX	$\lambda = 2, \quad \alpha = 2$
IV	$\lambda = 0, \quad \alpha = 0$	X	$\lambda = 3, \quad \alpha = 3$
V	$\lambda = 1, \quad \alpha = 2$	XI	$\lambda = 3, \quad \alpha = 4$
VI	$\lambda = 1, \quad \alpha = 2$	XII	$\lambda = 3, \quad \alpha = 4$

which proves (16).

This completes the proof of case (i) of Theorem 1. $\square$

We now give the special case $A = 0$ as a corollary to Theorem 1.

Corollary. *Let $K = Q(\theta)$ be a cyclic quartic extension of Q , where θ is a root of the irreducible polynomial $X^4 + BX + C$, where B and C are (nonzero) integers for which there does not exist a prime p with $p^3 \mid B$, $p^4 \mid C$. Then the conductor $f(K)$ of K is given by*

$$f(K) = 2^\delta \prod_{\substack{p \neq 2 \\ p \mid B, p \mid C}} p,$$

where the values of δ are given in Table (vi).

TABLE (vi): Values of δ		
δ	congruence conditions	examples
0	$B \equiv C \equiv 1 \pmod{2}$	$X^4 - 5X + 5$ $f(K) = 5$
2	$B \equiv 0 \pmod{8}, C \equiv 4 \pmod{8}$	$X^4 - 272X + 884$ $f(K) = 2^2 \cdot 17$
3	$B \equiv 0 \pmod{4}, C \equiv 1 \pmod{2}$	$X^4 - 20X + 95$ $f(K) = 2^3 \cdot 5$
4	$B \equiv 0 \pmod{8}, C \equiv 2 \pmod{4}$	$X^4 + 8X + 14$ $f(K) = 2^4$

PROOF. We first show that we cannot have

$$A = 0, B \equiv 0 \pmod{8}, C \equiv 0 \pmod{8}$$

in case (i) of the theorem. Suppose this possibility occurs. Then, by (1), we must have $C \equiv 8 \pmod{16}$, and, by Proposition 1, we have $S \equiv 1, 2$, or $5 \pmod{8}$. Define the integers r, s and x by

$$2^r \parallel R, 2^s \parallel S, 2^x \parallel B_1.$$

As S is squarefree we have $s = 0$ or 1 . From (4) (with $A = 0$) and (5) we obtain

$$2^{2r+s} \parallel t, \quad 2^{x+r+s} \parallel B.$$

As $B \equiv 0 \pmod{8}$ we must have

$$x + r + s \geq 3.$$

From (6) we have

$$4C = t^2 - B_1^2 S.$$

Note that $2^{4r+2s} \parallel t^2$ and $2^{2x+s} \parallel B_1^2 S$. We consider three cases

- (a) $4r + 2s < 2x + s$,
- (b) $4r + 2s = 2x + s$,
- (c) $4r + 2s > 2x + s$.

Case (a). In this case we have $2^{4r+2s} \parallel 4C$, so that $4r + 2s = 5$, which is impossible.

Case (b). In this case $4r + 2s = 2x + s \leq 5$ so that $s = 0$, $x = 2r$, $r = 0$ or 1 . If $r = 0$ then we have $x = 0$ contradicting $x + r + s \geq 3$. Hence we have $r = 1, x = 2, s = 0$, so that

$$2 \parallel R, S \equiv 1 \pmod{4}, 2^2 \parallel B_1, 2^2 \parallel t, 2^3 \parallel B, 2^3 \parallel C.$$

Setting

$$t = 4t_1, B_1 = 4B_2, C = 8C_1,$$

where t_1, B_2, C_1 are all odd, in $4C = t^2 - B_1^2 S$, and dividing by 2^4 , we obtain $2C_1 = t_1^2 - B_2^2 S$. Taking this equation modulo 4 we obtain

$$2 \equiv 2C_1 \equiv t_1^2 - B_2^2 S \equiv 1 - 1 \equiv 0 \pmod{4},$$

which is impossible.

Case (c). In this case we have $4r + s > 2x$ and $2^{2x+s} \parallel 4C$ so that $2x + s = 5$. Hence we have $s = 1$, $x = 2$ and $r \geq 1$. Thus we have

$$2^r \parallel R, S \equiv 2 \pmod{8}, 2^2 \parallel B_1, 2^{2r+1} \parallel t, 2^{r+3} \parallel B, 2^3 \parallel C.$$

Setting

$$t = 2^{2r+1}t_1, B_1 = 4B_2, C = 8C_1, S = 2S_1,$$

where $t_1 \equiv B_2 \equiv C_1 \equiv 1 \pmod{2}$, $S_1 \equiv 1 \pmod{4}$, in $4C = t^2 - B_1^2 S$, and dividing by 2^5 , we obtain $C_1 = 2^{4r-3}t_1^2 - B_2^2 S_1$. Taking this equation modulo 4 we obtain

$$C_1 \equiv \begin{cases} 2 - 1 \equiv 1 \pmod{4}, & \text{if } r = 1, \\ 0 - 1 \equiv 3 \pmod{4}, & \text{if } r \geq 2. \end{cases}$$

From (9) with $A = 0$ we have $16C - 3t^2 = Sz^2$, so that $S_1z^2 = 2^6C_1 - 3 \cdot 2^{4r+1}t_1^2$. If $r = 1$ then we have $2^5 \parallel S_1z^2$, which is impossible. Hence we have $r \geq 2$ and so $2^6 \parallel S_1z^2$, $2^6 \parallel z^2$, $2^3 \parallel z$, say $z = 2^3z_1$, where z_1 is odd. Thus $S_1z_1^2 = C_1 - 3 \cdot 2^{4r-5}t_1^2$. Taking this equation modulo 4 we obtain

$$1 \equiv S_1z_1^2 \equiv C_1 - 3 \cdot 2^{4r-5}t_1^2 \equiv 3 \pmod{4},$$

which is impossible.

This completes the proof that $B \equiv C \equiv 0 \pmod{8}$ does not occur when $A = 0$. The corollary now follows from case (i) of Theorem 1 with $A = 0$. $\square$

Our next two results give the unique quadratic subfield k (Theorem 2) and the discriminant $d(K)$ (Theorem 3) of the cyclic quartic field $K = Q(\theta)$, where $\theta^4 + A\theta^2 + B\theta + C = 0$, explicitly in terms of the prime factors of A, B and C .

Theorem 2. *With the notation of Theorem 1, the unique quadratic subfield of the cyclic quartic field $K = Q(\theta)$ where $\theta^4 + A\theta^2 + B\theta + C = 0$, is $k = Q(\sqrt{S})$, where S is given as follows:*

Case (i): $A^2 - 4C \neq 0, B \neq 0$.

$$S = 2^\theta \prod_{\substack{p \neq 2 \\ p|A, p|B, p|C \\ v_p(B) < v_p(A^2 - 4C)}} p \prod_{\substack{p \neq 2 \\ p|A, p|B, p|C \\ v_p(A^2 - 4C) \text{ odd} \leq v_p(B), v_p(C) \text{ odd}}} p,$$

where $\theta = 0$ except in the following cases when $\theta = 1$:

$$\begin{aligned} &A \equiv 4 \pmod{16}, \quad B \equiv 16 \pmod{32}, \quad C \equiv 28 \pmod{32}, \\ &\quad \ell = 5, \quad b = 4, \quad (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}, \\ &A \equiv 8 \pmod{16}, \quad B \equiv 0 \pmod{32}, \quad C \equiv 8 \pmod{32}, \\ &\quad \ell = 5, \quad b \geq 5, \quad (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}, \\ &A \equiv 12 \pmod{16}, \quad B \equiv 64 \pmod{128}, \quad C \equiv 4 \pmod{32}, \\ &\quad \ell \geq 10, \quad b = 6, \\ &A \equiv 12 \pmod{16}, \quad B \equiv 0 \pmod{256}, \quad C \equiv 4 \pmod{32}, \\ &\quad \ell(\text{odd}) = b + 3 \geq 11, \\ &A \equiv 0 \pmod{8}, \quad B \equiv 8 \pmod{16}, \quad C \equiv 6 \pmod{8}, \\ &\quad \ell = b = 3, \quad (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}, \\ &A \equiv 4 \pmod{8}, \quad B \equiv 0 \pmod{16}, \quad C \equiv 2 \pmod{8}, \\ &\quad \ell = 3, \quad b \geq 4, \quad (A^2 - 4C)/2^\ell \equiv 1 \pmod{4}, \\ &A \equiv 12 \pmod{16}, \quad B \equiv 32 \pmod{64}, \quad C \equiv 4 \pmod{32}, \\ &\quad \ell = 7, \quad b = 5, \quad (A^2 - 4C)/2^\ell \equiv 3 \pmod{4}, \\ &A \equiv 12 \pmod{16}, \quad B \equiv 0 \pmod{128}, \quad C \equiv 4 \pmod{32}, \\ &\quad \ell(\text{even}) = b + 3 \geq 10, \end{aligned}$$

where $\ell = v_2(A^2 - 4C)$ and $b = v_2(B)$.

Case (ii): $A^2 - 4C = 0, B \neq 0$.

$$S = 2^\phi \prod_{\substack{p \neq 2 \\ p|A, p^2 \parallel B}} p \prod_{\substack{p \neq 2 \\ p \parallel A, p^3 \mid B}} p,$$

where $\phi = 0$ except where $v_2(B) = 6$ in which case $\phi = 1$.

Case (iii): $A^2 - 4C \neq 0, B = 0$.

$$S = 2^\rho \prod_{\substack{p \neq 2 \\ v_p(C) \text{ odd}}} p,$$

where

$$\rho = \begin{cases} 0, & \text{if } v_2(C) \text{ even,} \\ 1, & \text{if } v_2(C) \text{ odd.} \end{cases}$$

PROOF. We just treat Case (i). By (8) we have $k = Q(\sqrt{S})$. From the tables immediately following (56), we see that the 2-part of S is 2^θ , where

$$\theta = \begin{cases} 0, & \text{in cases I–X,} \\ 1, & \text{in cases XI, XII.} \end{cases}$$

From Table (v), remembering that S is squarefree, we see that the odd part of S is

$$\prod_{\substack{p \neq 2 \\ p \mid A, p \mid B, p \mid C \\ v_p(B) < v_p(A^2 - 4C)}} p \quad \prod_{\substack{p \neq 2 \\ p \mid A, p \mid B, p \mid C \\ v_p(A^2 - 4C)(\text{odd}) \leq v_p(B) \\ v_p(C) \text{ odd}}} p.$$

This proves the asserted formula for S . $\square$

Before stating our next theorem, we recall that $\alpha, \beta, \gamma, \theta, \phi, \rho$ are defined in Table (i), Table (ii), Table (iii), Theorem 2 (Case (i)), Theorem 2 (Case (ii)), Theorem 2 (Case (iii)) respectively.

Theorem 3. *With the notation of Theorems 1 and 2, the discriminant $d(K)$ of the cyclic quartic field $K = Q(\theta)$, where $\theta^4 + A\theta^2 + B\theta + C = 0$, is given as follows:*

Case (i): $A^2 - 4C \neq 0, B \neq 0$.

$$d(K) = 2^{2\alpha+3\theta} \prod_{p \in S_2} p^2 \prod_{p \in S_3} p^3,$$

where

$$S_2 = \left\{ p \neq 2 \mid \begin{aligned} &v_p(B)(\text{odd}) < v_p(A^2 - 4C), \ p \nmid C \\ &\text{or } v_p(A^2 - 4C)(\text{odd}) \leq v_p(B), \ v_p(C) \text{ even} \\ &\text{or } 2 \leq v_p(A^2 - 4C)(\text{even}) \leq v_p(B), \ p \mid C \end{aligned} \right\}$$

and

$$S_3 = \left\{ p \neq 2 \mid \begin{aligned} &1 \leq v_p(B) < v_p(A^2 - 4C), \ p \mid C \\ &\text{or } v_p(A^2 - 4C)(\text{odd}) \leq v_p(B), \ v_p(C) \text{ odd} \end{aligned} \right\}.$$

Case (ii): $A^2 - 4C = 0, B \neq 0$.

$$d(K) = 2^{2\beta+3\phi} \prod_{p \in S_2} p^2 \prod_{p \in S_3} p^3,$$

where

$$S_2 = \left\{ p \neq 2 \mid p \parallel B \text{ or } p \nmid A, v_p(B)(\text{odd}) \geq 3 \right\},$$

and

$$S_3 = \left\{ p \neq 2 \mid p \mid A, p^2 \parallel B \text{ or } p \parallel A, p^3 \mid B \right\}.$$

Case (iii): $A^2 - 4C \neq 0, B = 0$

$$d(K) = 2^{2\gamma+3\rho} \prod_{p \in S_2} p^2 \prod_{p \in S_3} p^3,$$

where

$$S_2 = \left\{ p \neq 2 \mid p \mid A, v_p(C)(\text{even}) \geq 2 \right\},$$

and

$$S_3 = \left\{ p \neq 2 \mid v_p(C) \text{ odd} \right\}.$$

PROOF. This theorem follows from $d(K) = f(K)^2 d(k)$, $d(k) = 2^{2v_2(S)} S$, Theorem 1 and Theorem 2. $\square$

Our final theorem gives a necessary and sufficient condition for a cyclic quartic field to be totally imaginary.

Theorem 4. *With the notation of Theorem 1, let K be the cyclic quartic field $Q(\theta)$, where θ is a root of $\theta^4 + A\theta^2 + B\theta + C = 0$. Then*

Case (i): K is totally imaginary $\iff 2A^3 - 8AC + B^2 > 0$,

Case (ii): K is always totally imaginary,

Case (iii): K is totally imaginary $\iff A > 0$.

PROOF. We just treat Case (i). We have $K = Q(\sqrt{m + n\sqrt{S}})$. As K is cyclic we have $K = Q(\sqrt{m \pm |n|\sqrt{S}})$, and there exists an integer $k (\neq 0)$ such that $m^2 - Sn^2 = Sk^2$. Thus $|m| > |n|\sqrt{S}$. If $m > 0$ then $m > |n|\sqrt{S}$ so $m - |n|\sqrt{S} > 0$ and K is totally real. If $m < 0$ then

$-m > |n|\sqrt{S}$ so $m + |n|\sqrt{S} < 0$ and K is totally imaginary. We have thus shown that

$$K \text{ is totally imaginary} \iff m < 0.$$

By (10) we have

$$m < 0 \iff t + A > 0,$$

and, as $t + A$ is the unique real root of the polynomial

$$X^3 - 4AX^2 + (5A^2 - 4C)X + (-2A^3 + 8AC - B^2),$$

we have

$$t + A > 0 \iff -2A^3 + 8AC - B^2 < 0,$$

completing the proof. $\square$

We close by remarking that Theorem 5 of [1] follows easily from Theorem 1.

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