

On the closure of the basic subgroup.

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§ 1. Introduction.

A result of KULIKOV ([1], Theorem 11) states that an abelian p -group G containing no elements of infinite height is a direct summand of every abelian p -group in which G is a pure subgroup if and only if G is the closure of its basic subgroup. In this paper we prove that in the general case, i. e. if we omit the condition that G shall not contain elements of infinite height, a similar statement holds for the reduced part of G . More precisely, an arbitrary abelian p -group G is a direct summand of every abelian p -group in which G is a pure subgroup if and only if G can be decomposed into a direct sum of an algebraically closed group A and a group B^* which is the closure of a basic subgroup of G . We should like to note that even in the special case of groups without elements of infinite height, treated by KULIKOV, our proof seems to be simpler than the original one.

§ 2. Preliminaries.

By a group we shall always mean an additively written abelian group. (As every abelian torsion group admits a decomposition into the direct sum of its uniquely determined primary components, the latter being p -groups — i. e. groups every element of which is of p -power order, where p denotes a fixed prime number — our considerations can be extended to arbitrary abelian torsion groups in a natural way.) Let G be a group, $O(g)$ the order of g ($g \in G$), nG the set of all elements of the form ng (n is a rational integer, $g \in G$).

If there exists an element of maximal order r in G , then we say that G is an r -bounded group. The r -bounded group G will be called regular if

$$g \in \frac{r}{O(g)} G$$

for each element $g \in G$. In his paper [2] T. SZELE has proved the following

Lemma 1. *A regular r -bounded subgroup H of an arbitrary abelian group G is a direct summand of G if and only if $H \cap rG = 0$. Moreover, if H is a regular r -bounded direct summand of G , then any subgroup K of G , which is maximal among the subgroups of G having zero intersection with H and containing rG , is a direct complement of H , i. e. $G = H + K$.*

Let H be a subgroup of G . If for each $a \in H$ the solvability of $nx = a$ in G implies its solvability in H , then H is said to be a pure subgroup of G .

An abelian group A is called algebraically closed (in an other terminology divisible or complete) if for each $a \in A$ and every positive rational integer n the equation $nx = a$ is solvable in A . It is well-known that every abelian group is the direct sum of its (uniquely determined) maximal algebraically closed subgroup and of a reduced group, i. e. a group without non-zero algebraically closed subgroups. The latter is often mentioned as the reduced part of the group (though it is determined merely up to isomorphism).

The height of an element $a \neq 0$ in G is defined as follows. For a nonzero element a of the p -group G the maximal non-negative integer k for which the equation $p^k x = a$ is solvable in G is said to be the height of a . If there is no maximal k with this property, then a has infinite height.

Let G be a p -group and B_n ($n = 1, 2, \dots$) a maximal p^n -bounded regular direct summand of G_{n-1} ($n = 1, 2, \dots$), where $G_0 = G$ and G_{n-1} is the complemented direct summand of B_{n-1} (the existence of such subgroups follows by Lemma 1 and by Zorn's lemma). Then

$$G = B_1 + G_1 = B_1 + B_2 + G_2 = \dots = B_1 + B_2 + \dots + B_n + G_n = \dots,$$

and $B_1 + B_2 + \dots + B_n$ is a maximal p^n -bounded direct summand of G . It has been proved by T. SZELE in [2] that $B = B_1 + B_2 + \dots + B_n + \dots$ is a basic subgroup of G in the sense of KULIKOV [1], i. e. B is a direct sum of cyclic groups and is a pure subgroup in G , while G/B is an algebraically closed group. The torsion subgroup B^* of the complete direct sum of the B_n 's is called the closure of B . B^* can be considered as the set of all infinite „vectors“

$$\langle b_1, b_2, \dots, b_n, \dots \rangle \quad (b_1 \in B_1, b_2 \in B_2, \dots, b_n \in B_n, \dots)$$

(with exactly one component from each B_n) having finite order with respect to the addition defined component-wise. KULIKOV has proved that B is a basic subgroup of B^* and that any group without elements of infinite height can be considered as a pure subgroup of the closure of its basic subgroup.

§ 3. Groups with Property P .

In what follows we turn to the description of all p -groups which have the following

PROPERTY P . G is a direct summand of any p -group in which it is a pure subgroup.

First we prove a Lemma which seems to be of some interest in itself, though we need only a part of it in the proof of our Theorem.

Lemma 2. *An abelian p -group G has Property P if and only if every proper direct summand of G has Property P .*

PROOF. Let G be a group with Property P and $G = A + C$ an arbitrary direct decomposition of G . We have to show that, e. g., if A is a pure subgroup in H , then A is a direct summand of H . Let us take a group C' isomorphic to C and having no nonzero elements in common with H , then $G' = A + C'$ also has Property P . Considering $K = C' + H$, it follows easily that G' is pure in K and therefore $K = G' + D = A + C' + D$. Now $H = A + [H \cap (C' + D)]$ proves our first statement.

Conversely, suppose that every proper direct summand of G has Property P and $G = A + B$ ($A \neq 0, B \neq 0$). (If G has no proper direct summands then it is either a cyclic or a quasicyclic group and so has Property P .) Let G be a pure subgroup in H . We shall prove that G is a direct summand of H . Since A too is a pure subgroup in H and has Property P , there must be a direct decomposition $H = A + C$. Now $G = A + (C \cap G)$ shows that $C \cap G$ ($\cong B$) is also a group with Property P and is pure in C ; for a suitable subgroup D of C this gives $C = (C \cap G) + D$. Thus we have $H = A + C = A + (C \cap G) + D = G + D$, which completes the proof of Lemma 2.

Now we are ready to state our

Theorem. *An abelian p -group G has Property P if and only if G can be decomposed into the direct sum $G = A + B^*$ of an algebraically closed group A and of the closure B^* of a basic subgroup B of G .*

PROOF. *The condition is necessary.* As a first step we show that if a group G with Property P contains a nonzero element of infinite height, then G has a nonzero algebraically closed subgroup.

Let Q be a quasicyclic group, i. e. a group which is generated by the elements $c_1, c_2, \dots, c_n, \dots$ and is defined by the relations $pc_1 = 0, pc_2 = c_1, \dots, pc_n = c_{n-1}, \dots$, and denote by H the direct sum of G and Q : $H = G + Q$. Suppose that $g_0 \in G$ is an element of order p and of infinite height in G , so that there exist elements $g_1, g_2, \dots, g_n, \dots$ ($\in G$) such that $g_n = pg_{n-1} =$

$= p^2 g_2 = \dots = p^n g_n = \dots$, and let us consider the factor group $\bar{H} = H/\{g_0 - c_1\}$. Since $G \cap \{g_0 - c_1\} = Q \cap \{g_0 - c_1\} = 0$, there exist in $\bar{H}$ subgroups $\bar{G} \cong G$ and $\bar{Q} \cong Q$ which are the images of G resp. Q under the natural homomorphism $H \sim \bar{H}$. We show that $\bar{G} \cap \bar{Q} = \{\bar{g}_0\} = \{\bar{c}_1\}$ ¹⁾. From $\overline{g_0 - c_1} = 0$ it follows that $\bar{g}_0 = \bar{c}_1$; supposing that $\bar{g} = \bar{q}$, where $\bar{g} \in \bar{G}$ and $\bar{q} \in \bar{Q}$ we obtain $\overline{g - q} = 0$ and hence $g - q = kg_0 - kc_1$, i. e. $g - kg_0 = q - kc_1$. Since $Q \cap G = 0$ we can infer that $g = kg_0$ resp. $q = kc_1$. This shows that $\bar{g} \in \{\bar{g}_0\}$ and $\bar{q} \in \{\bar{c}_1\}$, but $\bar{c}_1 = \bar{g}_0$ and so $\bar{G} \cap \bar{Q} = \{\bar{g}_0\} = \{\bar{c}_1\}$ as desired. Let us now imbed G and Q in $\bar{H}$ with the aid of the identifications $\bar{g} = g$ ($g \in G$) and $\bar{q} = q$ ($q \in Q$), respectively. In the sequel no distinction will be made between $\bar{G}$ and G , and between $\bar{Q}$ and Q . Then $\bar{H} = \{G, Q\}$ and $G \cap Q = \{g_0\} = \{c_1\}$. Now G is a pure subgroup in $\bar{H}$ since $g = p^k h = p^k(g' + rc_n)$ ($g, g' \in G, h \in \bar{H}$ and r is a rational integer) implies $p^k rc_n = g - p^k g' \in (Q \cap G) = \{g_0\}$ and so for a suitable integer s we have $p^k rc_n = sg_0 = sp^k g_k$ which gives $g = p^k(g' + sg_k)$. Thus $\bar{H} = G + K$, which implies $c_1 = g_0, c_2 = g'_2 + f_2, \dots, c_n = g'_n + f_n, \dots$ ($f_2 \in K, \dots, f_n \in K, \dots$) and it follows that $\{g_0, g'_2, \dots, g'_n, \dots\}$ is a nonzero quasicyclic subgroup of G .

Now let us consider an arbitrary group G with Property P . G has a decomposition $G = A + C$ where A is the maximal algebraically closed subgroup of G and C is a reduced group. Thus, by Lemma 2, C has Property P and by the statement proved above it can have no nonzero elements of infinite height. So we merely have to refer to KULIKOV's result that a group C without nonzero elements of infinite height can be considered as a pure subgroup of the closure B^* of its basic subgroup B and to the fact that C cannot be a proper direct summand of B^* . From this it follows immediately that $C = B^*$, and since a basic subgroup B of C is also a basic subgroup of G , this is the statement which was to be proved.

The condition is sufficient. First we prove that if the group G is the closure of its basic subgroup B then G has Property P . We prove this by making use of the construction $B = B_1 + B_2 + \dots + B_n + \dots$ of the basic subgroup given by T. SZELE, where $G = B_1 + G_1, G_1 = B_2 + G_2, \dots, G_{n-1} = B_n + G_n, \dots$. Let G be a pure subgroup in H . We shall see by induction that $H = B_1 + H_1, H_1 = B_2 + H_2, \dots, H_{n-1} = B_n + H_n, \dots$. If we take $B_0 = 0, G_0 = G$ and $H_0 = H$, then we have $H = B_0 + H_0, G_0 \subseteq H_0$ and so we can proceed to the inductive step at once. Supposing $H = B_1 + H_1, H_1 = B_2 + H_2, \dots, H_{n-2} = B_{n-1} + H_{n-1}$ and $G_n \subseteq H_n$ we have to prove $H_{n-1} = B_n + H_n$ and $G_n \subseteq H_n$. Since B_n is a p^n -bounded regular subgroup of H_{n-1} (as $B_n + G_n = G_{n-1} \subseteq H_{n-1}$),

¹⁾ If $h \in H$, then we denote by $\bar{h}$ the image of h under the natural homomorphism $H \sim \bar{H}$.

Lemma 1 shows that it is sufficient to prove $\{G_n, p^n H_{n-1}\} \cap B_n = 0$. Consider $b = g + p^n h$ ($b \in B_n, g \in G_n, h \in H_{n-1}$). As $p^n h = b - g \in (B_n + G_n) = G_{n-1}$ and G_{n-1} is pure in H_{n-1} , there exists an element $b' + g'$ ($b' \in B_n, g' \in G_n$) in G_{n-1} , for which $p^n(b' + g') = p^n h$. But $p^n b' = 0$ and so we have $b = g + p^n g' \in (B_n \cap G_n) = 0$, and hence $b = 0$.

Thus we can get for every $h \in H$ the decompositions

$$h = b_1 + h_1 = b_1 + b_2 + h_2 = \dots = b_1 + b_2 + \dots + b_n + h_n = \dots$$

$$(b_1 \in B_1, h_1 \in H_1, b_2 \in B_2, h_2 \in H_2, \dots, b_n \in B_n, h_n \in H_n, \dots).$$

It is easy to see that the mapping $h \rightarrow \langle b_1, b_2, \dots, b_n, \dots \rangle$ (where $\langle b_1, b_2, \dots, b_n, \dots \rangle$ is an element of $B^* = G$) is a projection of H onto G (i. e. a homomorphism under which the elements of G remain invariant) and this proves that G is a direct summand of H .

As to the general case, let $G = A + B^*$ be a pure subgroup in H , where A is an algebraically closed group and B^* is the closure of a basic subgroup of G . Since A is algebraically closed, H can be decomposed into a direct sum $H = A + C$ such that $B^* \subseteq C$. B^* is pure in C , so by what we have proved above B^* is a direct summand of C , i. e., $C = B^* + D$. Hence $H = A + C = A + B^* + D = G + D$ and this completes the proof of the Theorem.

In view of the fact that a group of bounded order is the closure of its basic subgroup, we get the following

Corollary. Every p -group of bounded order has Property P .

Bibliography.

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(Received August 1, 1957.)