On the splitting problem of mixed abelian groups.

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§ 1. Introduction.

The theory of abelian groups is usually divided into three great parts: theory of torsion groups, torsion-free groups and mixed groups.[1]) The basic problem concerning mixed groups is: how the study of mixed groups could be reduced to the examination of torsion groups and of torsion-free groups? This is frequently stated in the following special form: under what conditions decomposes a mixed group into a direct sum of a torsion group and a torsion-free group? Making further distinctions we arrive at the following three problems:

(I) For which torsion groups T does the relation $\mathrm{Ext}(T, H) = 0$ hold with every torsion-free group H?[2])

(II) For which torsion-free groups H does the relation $\mathrm{Ext}(T, H) = 0$ hold with every torsion group T?

(III) What conditions must be satisfied by a torsion group T and by a torsion-free group H in order that the relation $\mathrm{Ext}(T, H) = 0$ be valid?

Up to now only the first of these problems had been fully settled. Namely, a torsion group satisfies the condition of (I) if and only if it is a direct sum of a group of bounded order and a divisible group (R. BAER [1], S. V. FOMIN [3]). The remaining two problems are open, only partial results are known. We mention here a partial answer to the second one: a countable torsion-free group satisfies the condition of (II) if and only if it is a free group (R. BAER [1]).

In the present paper we shall give a new approach to these problems, and in this way we obtain results concerning extensions of p-groups by

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[1]) *In what follows by a group we shall mean always an additively written abelian group.* For the fundamentals of the theory of groups see e. g. A. G. KUROS [8] and I. KAPLANSKY [5]. Numbers in brackets refer to the bibliography at the end of the paper.

[2]) $\mathrm{Ext}(A, B) = 0$ expresses the following property of the groups A and B: any extension of A by B contains A as a direct summand.

torsion-free groups (Theorem 9 and Theorem 10). By virtue of these we can solve the second problem in the case when the torsion-free groups under consideration have countable p-adic dimension for at least one p (Theorem 11).[3]) By means of our theorems it is possible to give answer to an open question, raised by R. BAER in 1936. It is asked whether the group of all sequences of rational integers satisfies the condition of the second problem above (R. BAER [1]; see also I. KAPLANSKY [5]). The answer is negative (§ 4).

A method of examination of torsion groups and torsion-free groups is based on the fact that any p-group can be regarded as a module over the ring of p-adic integers and any torsion-free group can be embedded in such a module; in this way one obtains results for these groups by studying p-adic modules.[4]) Now we set as an aim the investigation of the above mentioned splitting problems of mixed groups by the methods of p-adic modules. This is motivated by the relatively advantageous algebraic and topological properties of p-adic modules.[5]) Our main theorems are contained in § 4, but we get over the main difficulties in § 2; the connection between the extensions of groups and that of p-adic modules is established in § 3.

There are several ways of generalizing our propositions; e. g. by changing the class of groups, or the operator domain of modules, or the topological structure, etc.

§ 2. Extensions of p-adic modules.

In this section we settle the splitting problems of mixed p-adic modules, supposing that the torsion-free modules under consideration are of countable dimension.[6]) Throughout we shall make use of Theorem 1 which seems to be of some interest also in itself. Lemma 2 is a consequence of a theorem due to S. EILENBERG and S. MACLANE [2] (cf. also R. BAER [1] and E. SASIADA [9]). The crucial point of the whole paper is the proof of Lemma 3. The final conclusions of this § are Theorem 4 and Theorem 5.

Theorem 1. *Let G be a p-adic module without elements of infinite height and let be given a direct decomposition of a basic submodule of G*

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[3]) For the definition of the p-adic dimension of groups see § 4.

[4]) For the theory of modules over the ring of p-adic integers (in the sequel p-adic modules or briefly modules) see I. KAPLANSKY [5].

[5]) Any p-adic module having no elements of infinite height can be equipped with a topology (p-adic topology) in a natural way: the submodules $p^nG (n = 1, 2, \ldots)$ form a complete system of neighbourhoods of 0.

[6]) The concept of the dimension of a p-adic module will be introduced in the present §. This is suggested by the analogy between Hilbert spaces and p-adic modules established in Theorem 1.

into a direct sum of cyclic submodules B_ν $(\nu \in \Gamma)$.[7]) *Then any element of* G *can be represented as the sum of a convergent infinite series the non-zero terms of which belong to different* B_ν *s.*[8]) *This representation is unique apart from the order of the members of the series and apart from* 0 *terms.*

PROOF. First of all we remark that the B_ν s and the submodule $B = \sum_{\nu \in \Gamma} B_\nu$ inherit their p-adic topology from the p-adic topology of G, therefore the convergence of sequences and series is understood always in the sense of the p-adic topology of G.

The submodule B is an everywhere dense subset of G, consequently any element $g \in G$ can be represented as the sum of a convergent infinite series $a_1 + \cdots + a_n + \cdots$ the terms of which belong to B. Denote by $a_{n\nu}$ the component of a_n in B_ν. It is easy to see that the series $a_{1\nu} + \cdots + a_{n\nu} + \cdots$ converges to an element b'_ν of B_ν for each $\nu \in \Gamma$. The convergence of the series $a_1 + \cdots + a_n + \cdots$ implies that $a_1, \ldots, a_n, \ldots$ is a 0-sequence, thus $a_{1\nu}, \ldots, a_{n\nu}, \ldots$ also tends to 0; since B_ν is a complete module, $a_{1\nu} + \cdots + a_{n\nu} + \cdots$ has a sum b'_ν in B_ν. The subset of all non-zero b'_ν s is countable. Indeed, all components of any a_n are 0 except for a finite number of indices $\nu \in \Gamma$, so $a_{n\nu} \neq 0$ holds only for a countable set of pairs n, ν (n is a positive integer, $\nu \in \Gamma$), i. e. all terms of the series $a_{1\nu} + \cdots + a_{n\nu} + \cdots$ are 0 with the exception of a countable set of indices $\nu \in \Gamma$. Now let $b_1 + \cdots + b_n + \cdots$ be a series formed by all non-zero b'_ν s in an arbitrary order and possibly completed by zeros. We show that this series converges to g. Let $k \geqq 0$ be an integer. There exists an index N with the property that for any $n \geqq N$ $a_1 + \cdots + a_n - g \in p^k G$, because $a_1 + \cdots + a_n + \cdots = g$. Evidently $a_n \in p^k G$ if $n > N$, hence $a_{n\nu} \in p^k G$ $(n > N, \nu \in \Gamma)$. Thus we have $a_{1\nu} + \cdots + a_{N\nu} - b'_\nu \in p^k G$ for any $\nu \in \Gamma$. Let $a_1 + \cdots + a_N \in B_{\nu_1} + \cdots + B_{\nu_r}$; it is clear that $b'_\nu \in p^k G$ if $\nu \neq \nu_i$ $(i = 1, \ldots, r)$. If m is a number large enough then the non-zero members of $b'_{\nu_1}, \ldots, b'_{\nu_r}$ occure among $b_1, \ldots, b_m$; therefore we get

$$b_1 + \cdots + b_m - g = b_1 + \cdots + b_m - (a_1 + \cdots + a_N) + (a_1 + \cdots + a_N) - g =$$
$$= b + b'_{\nu_1} + \cdots + b'_{\nu_r} - [(a_{1\nu_1} + \cdots + a_{1\nu_r}) + \cdots + (a_{N\nu_1} + \cdots + a_{N\nu_r})] +$$
$$+ (a_1 + \cdots + a_N) - g = b + [b'_{\nu_1} - (a_{1\nu_1} + \cdots + a_{N\nu_1})] + \cdots +$$
$$+ [b'_{\nu_r} - (a_{1\nu_r} + \cdots + a_{N\nu_r})] + a_1 + \cdots + a_N - g \in p^k G$$

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[7]) A pure submodule B of G is a basic submodule if it decomposes into a direct sum of cyclic modules and G/B is divisible. Any p-adic module has basic submodules and all these are isomorphic. For the concept and properties of basic submodules see L. YA. KULIKOV [6], [7], L. KALOUJNINE [4], T. SZELE [10].

[8]) The convergence of series is understood in the sense of the p-adic topology of G.

(here b denotes the sum of those ones of $b_1, \ldots, b_m$ which differ from $b'_{\nu_1}, \ldots, b'_{\nu_r}$). With this we proved $b_1 + \cdots + b_n + \cdots = g$.

In order to prove the uniqueness let us consider representations

$$g = b_1^{(1)} + \cdots + b_n^{(1)} + \cdots \quad \text{and} \quad g = b_1^{(2)} + \cdots + b_n^{(2)} + \cdots$$

of $g \in G$, where in both series the non-zero terms belong to different B_ν s. We may suppose that $b_n^{(1)}$ and $b_n^{(2)}$ belong to the same B_ν for each index n; this can be done always by reordering the series and by completion with zeros. To show this the following remark is sufficient: if $g_1 + \cdots + g_n + \cdots$ is a convergent series of elements of G then any series obtained by reordering from the former one converges also to the same element of G. We shall omit the proof of this simple fact. Now from the above representations of g we get

$$(b_1^{(1)} - b_1^{(2)}) + \cdots + (b_n^{(1)} - b_n^{(2)}) + \cdots = 0.$$

Thus for any integer $k \geqq 0$ one has

$$(b_1^{(1)} - b_1^{(2)}) + \cdots + (b_n^{(1)} - b_n^{(2)}) \in p^k G$$

if n is large enough. This implies $b_n^{(1)} = b_n^{(2)}$ for every index n, from which the desired uniqueness follows.

Lemma 2. *Let A' be a factor module of the p-adic module A, and B' a submodule of the p-adic module B. Then* $\mathrm{Ext}_p(A, B) = 0$ *implies* $\mathrm{Ext}_p(A', B') = 0$.[9])

PROOF. Let B be represented as a factor module F/F_0 of a free module F; the elements of F/F_0 will be indentified with the elements of B. Let $F' \subseteq F$ be the submodule for which $F'/F_0 = B'$ holds. The condition $\mathrm{Ext}_p(A, B) = 0$ is equivalent to the following one: every homomorphism of F_0 into A can be extended to a homomorphism of F into A (the corresponding statement for groups is a simple consequence of a theorem of S. EILENBERG and S. MACLANE [2]; one proves the analogue of this corollary for p-adic modules by similar considerations). Accordingly, in order to prove the relation $\mathrm{Ext}_p(A', B') = 0$ it is sufficient to show that any homomorphism φ of F_0 into A' can be extended to a homomorphism φ' of the free module F' into A'. Let x_ν $(\nu \in \Gamma)$ be a basis of the free module F_0 and choose a representative $a_\nu \in A$ from each coset $x_\nu \varphi$ of the module A. The correspondence $x_\nu \to a_\nu$ $(\nu \in \Gamma)$ defines a homomorphic mapping ψ of F_0 into A. The condition $\mathrm{Ext}_p(A, B) = 0$ implies that ψ can be extended to a homomorphic mapping of F into A, and a

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[9]) $\mathrm{Ext}_p(A, B) = 0$ expresses the analogous property of the p-adic modules A and B as $\mathrm{Ext}(A, B) = 0$ for groups, but extensions are understood in module-theoretical sense (see footnote 2).

fortiori to that of F' into A; this will be denoted by ψ'. The mapping ψ' induces a homomorphism φ' of F into A'. It is clear that φ' continues φ; thus, indeed $\mathrm{Ext}_p(A', B') = 0$.

Definition. *By the dimension of a p-adic module we mean the (common) rank of its basic submodules.*[6, 7])

Lemma 3. *Let H be a p-adic torsion-free module of countable dimension and $T = \sum_n C(p^n)$. Then $\mathrm{Ext}_p(T, H) = 0$ implies that H is a free module.*[10])

PROOF. First of all we show that H does not contain elements of infinite height (0 is excluded of course). Suppose the contrary. Then H has a divisible torsion-free submodule A of rank 1, because H is a torsion-free module. By Lemma 2 $\mathrm{Ext}_p(T, H) = 0$ implies $\mathrm{Ext}_p(T, A) = 0$. The completion T^* of the module T contains elements of infinite order, for example the sum of the series $\sum_n p^n c_{2n}$ where c_m is a generator of the cyclic module $C(p^m)$ occuring in the decomposition $T = \sum_n C(p^n)$. It follows that T^* has submodules T' and T'' containing T for which

$$T^*/T = (T'/T) + (T''/T)$$

and $T'/T \cong A$ hold, since T^*/T is a divisible module. Now by $\mathrm{Ext}_p(T, A) = 0$, T is a direct summand of T': $T' = T + A'$. Thus T^* contains a divisible submodule $A' \cong A$. But this is impossible for T^* is a module without elements of infinite height. So indeed, H has no elements of infinite height

Let B be a basic submodule of H and $\overline{\varphi}$ an arbitrary homomorphic mapping of H/B into T^*/T. We shall prove that there exists a homomorphic mapping φ of H into T^* which induces $\overline{\varphi}$, i. e. $\overline{h\varphi} = \overline{h}\,\overline{\varphi}$ holds for every element $h \in H$ (bars indicate cosets of T^* modulo T or those of H modulo B according as the element under consideration belongs to T^* or to H). The mapping φ will be constructed by introducing the following module G. Let us choose for each element $h \in H$ an element $t_h^* \in T^*$ satisfying the relation $\overline{t_h^*} = \overline{h}\,\overline{\varphi}$; if $h \in B$ then we set $t_h^* = 0$. Let G be the subset of $T^* + H$ consisting of all elements of the form $t + t_h^* + h$ $(t \in T, h \in H)$. G is a submodule of $T^* + H$. Indeed, if $g_1 = t_1 + t_{h_1}^* + h_1$ and $g_2 = t_2 + t_{h_2}^* + h_2$ are arbitrary elements of G and r_1, r_2 are p-adic integers then by making use of the relation

$$\overline{r_1 t_{h_1}^* + r_2 t_{h_2}^*} = r_1 \overline{t_{h_1}^*} + r_2 \overline{t_{h_2}^*} = r_1(\overline{h_1}\,\overline{\varphi}) + r_2(\overline{h_2}\,\overline{\varphi}) = \overline{r_1 h_1 + r_2 h_2}\,\overline{\varphi} = \overline{t_{r_1 h_1 + r_2 h_2}^*}$$

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[10]) The cyclic group of order p^n (denoted by $C(p^n)$) is considered here as a p-adic module.

we get

$$r_1g_1+r_2g_2=r_1(t_1+t^*_{h_1}+h_1)+r_2(t_2+t^*_{h_2}+h_2)=$$
$$=(r_1t_1+r_2t_2)+(r_1t^*_{h_1}+r_2t^*_{h_2})+(r_1h_1+r_2h_2)=$$
$$=(r_1t_1+r_2t_2+t)+t^*_{r_1h_1+r_2h_2}+(r_1h_1+r_2h_2)\in G$$

where $t\in T$. It is obvious that $T\subseteqq G$. Moreover $G/T\cong H$, because the mapping $t+t^*_h+h\to h$ $(t\in T, h\in H)$ is a homomorphism of G onto H with kernel T. Therefore by the condition $\mathrm{Ext}_p(T,H)=0$ we conclude that T is a direct summand of G: $G=T+H'$. Clearly, $B\subseteqq G$. Let $b_1,\ldots,b_n,\ldots$ be a basis of B and $b'_1,\ldots,b'_n,\ldots$ be the sequence of those elements of H' for which $b'_n-b_n=t_n\in T$ hold. By virtue of Theorem 1 any element of H can be represented as the sum of a convergent series $r_1b_1+\cdots+r_nb_n+\cdots$, where $r_1,\ldots,r_n,\ldots$ is a 0-sequence of p-adic integers. Let the element $r_1t_1+\cdots+r_nt_n+\cdots$ of T^* correspond to the element $r_1b_1+\cdots+r_nb_n+\cdots$ of H; this is a homomorphic mapping φ of H into T^* ($r_1t_1+\cdots+r_nt_n+\cdots$ is a convergent series since $r_1,\ldots,r_n,\ldots$ is a 0-sequence of p-adic integers and T^* is a complete module). We show that φ induces $\overline{\varphi}$. The homomorphism $t+t^*_h+h\to h$ $(t\in T, h\in H)$ determines an isomorphism of H' onto H under which the image of b'_n is b_n. Therefore if $r_1b_1+\cdots+r_nb_n+\cdots$ has a sum h in H then $r_1b'_1+\cdots+r_nb'_n+\cdots$ is a convergent series in H'. Thus, taking into account that the p-adic topology of a submodule is at least so strong as its relative topology, we get

$$h\varphi+h=(r_1t_1+\cdots+r_nt_n+\cdots)+(r_1b_1+\cdots+r_nb_n+\cdots)=$$
$$=r_1(t_1+b_1)+\cdots+r_n(t_n+b_n)+\cdots=r_1b'_1+\cdots+r_nb'_n+\cdots\in H'\subseteqq G,$$

where the convergence of the series is understood in the sense of the p-adic topology of T^*+H. Hence by $t^*_h+h\in G$ we have $h\varphi-t^*_h=(h\varphi+h)-(t^*_h+h)\in G$. On the other hand by $h\varphi\in T^*$ we have $h\varphi-t^*_h\in T^*$. But these can hold simultaneously only in the case when $h\varphi-t^*_h\in T$, since, as it is easy to see, any common element of G and T^* belongs to T. From $\overline{h\varphi}=\overline{t^*_h}$ and $\overline{t^*_h}=\overline{h}\,\overline{\varphi}$ we infer $\overline{h\varphi}=\overline{h}\,\overline{\varphi}$; thus indeed, φ induces $\overline{\varphi}$.

Finally we prove that H is a free module. First we show that the set of all homomorphic mappings $\overline{\varphi}$ of the module H/B into T^*/T has cardinality $\geqq 2^{\mathfrak{m}}$, where $\mathfrak{m}$ is the rank of H/B. The module H/B is divisible and torsion-free, consequently H has submodules H_ν containing B (ν ranges over a set Γ of indices having cardinality $\mathfrak{m}$) for which

$$H/B=\sum_{\nu\in\Gamma}(H_\nu/B)$$

and $H_\nu/B\cong A$ $(\nu\in\Gamma)$ hold. Let us consider for each $\nu\in\Gamma$ either an isomorphic mapping of H_ν/B onto $T'/T\cong A$ or the 0 homomorphism of H_ν/B

into T^*/T; in this way we get a homomorphism of H/B into T^*/T. The set of all homomorphisms of this kind and a fortiori the set of all $\overline{\varphi}$ s has cardinality $\geqq 2^{\mathfrak{m}}$. On the other hand the set of all homomorphisms φ of the module H into T^* which induce the $\overline{\varphi}$ s is of cardinality $\leqq 2^{\aleph_0}$. In order to prove this we have only to remark that $B\varphi \subseteq T$ and φ is determined uniquely by the sequence $b_1\varphi, \ldots, b_n\varphi, \ldots$ of elements of T (since B is an everywhere dense subset of H); the set of these sequences is of cardinality $\leqq \aleph_0^{\aleph_0} = 2^{\aleph_0}$. Now it is obvious that the induced mapping $\overline{\varphi}$ is determined uniquely by φ, thus we have $2^{\mathfrak{m}} \leqq 2^{\aleph_0}$, i. e. $\mathfrak{m} \leqq \aleph_0$. Therefore H, as a module of countable dimension, is countably generated. From this it follows that H is a free module because H is a torsion-free module and has no elements of infinite height.

This completes the proof of our lemma.

Theorem 4. *Let H be a p-adic torsion-free module of countable dimension and T a p-adic torsion module. Then $\mathrm{Ext}_p(T, H) = 0$ holds if and only if at least one of the following two conditions is satisfied.*

(1) *T is a direct sum of a module of bounded order and a divisible module.*

(2) *H is a free module.*

PROOF. Suppose $\mathrm{Ext}_p(T, H) = 0$ and at the same time condition (1) is not satisfied. Then the basic submodule B of T is not of bounded order. Indeed, in the contrary case T would be the direct sum of B and a divisible module, as B is a pure submodule of T and T/B is divisible. So $\sum_n C(p^n)$ is a homomorphic image of B. On the other hand B is a homomorphic image of T (cf. T. SZELE [10]), thus $\sum_n C(p^n)$ is a homomorphic image of T. Making use of Lemma 2 we have $\mathrm{Ext}_p(\sum_n C(p^n), H) = 0$. This implies by virtue of Lemma 3 that H is a free module, so condition (2) is satisfied.

Conversely, if one of the conditions (1), (2) holds then $\mathrm{Ext}_p(T, H) = 0$, as it is well known.

This completes the proof.

Theorem 5. *Let H be a p-adic torsion-free module of countable dimension. Then $\mathrm{Ext}_p(T, H) = 0$ holds with every p-adic torsion module T if and only if H is a free module.*

The proof is clear by Theorem 4.

§ 3. p-adic closure of groups.

In this section the p-adic closure of groups is defined; the uniqueness and the condition for the existence are established in Theorem 6 and in Theorem 7. By Theorem 8 we can carry over the results of § 2 without difficulty to groups.

Definition. *A p-adic module $G^{(p)}$ containing the group G is a p-adic closure of G if the following two conditions are satisfied:*

(1) *$G^{(p)}$ is generated by the group G.*

(2) *Any independent subset of the group G is an independent subset of the module $G^{(p)}$.* [11])

Theorem 6. *Let $G_1^{(p)}$ resp. $G_2^{(p)}$ be a p-adic closure of the group G_1 resp. G_2. Then any isomorphism of the group G_1 onto G_2 can be extended in one and only one way to an isomorphism of the p-adic module $G_1^{(p)}$ onto $G_2^{(p)}$.*

Proof. Let φ be an isomorphism of G_1 onto G_2. First let G_1 be a finitely generated group. Then by virtue of the fundamental theorem of finitely generated groups, G_1 has a basis $a_1, \ldots, a_n$, and so $a_1\varphi, \ldots, a_n\varphi$ is a basis of G_2. According to the definition of p-adic closure, $a_1, \ldots, a_n$ is a basis of the module $G_1^{(p)}$ and $a_1\varphi, \ldots, a_n\varphi$ is a basis of $G_2^{(p)}$. It is easy to see that the correspondence $a_i \to a_i\varphi$ $(i = 1, \ldots, n)$ determines an isomorphic mapping of $G_1^{(p)}$ onto $G_2^{(p)}$. It is clear that this is the only possible extension of φ.

Now let G_1 be an arbitrary group. Any subgroup A_1 of G_1 generates in $G_1^{(p)}$ its p-adic closure $A_1^{(p)}$, since $G_1^{(p)}$ is a p-adic closure of G_1; similarly, $A_2 = A_1\varphi$ generates in $G_2^{(p)}$ its p-adic closure $A_2^{(p)}$. If A_1 is a finitely generated group then from the formerly treated special case of our proposition we conclude that there exists one and only one isomorphism φ_{A_1} of $A_1^{(p)}$ onto $A_2^{(p)}$ which is equal to φ on A_1. Throughout the proof of this theorem A_1 denotes a finitely generated subgroup of G_1. Now let us define a mapping φ^* in the following way: if g is an element of a submodule $A_1^{(p)} \subseteqq G_1^{(p)}$ then $g\varphi^* = g\varphi_{A_1}$. We show that φ^* is the required extension of φ. The mapping φ^* is defined for every element of $G_1^{(p)}$ since G_1 generates $G_1^{(p)}$. Furthermore φ^* is a single valued mapping. Indeed, if g belongs to $A_1'^{(p)} \subseteqq G_1^{(p)}$ and to $A_1''^{(p)} \subseteqq G_1^{(p)}$ then, denoting by A_1 the subgroup of G_1 generated by A_1' and A_1'', the image of g under $\varphi_{A_1'}$ and $\varphi_{A_1''}$ is $g\varphi_{A_1}$ since by the uniqueness of $\varphi_{A_1'}$ and $\varphi_{A_1''}$ the isomorphism φ_{A_1} continues $\varphi_{A_1'}$ and

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[11]) S is an independent subset of the group (p-adic module) G if any relation $r_1a_1 + \cdots + r_na_n = 0$, where $a_1, \ldots, a_n$ are different elements of S and $r_1, \ldots, r_n$ are rational (p-adic) integers, implies $r_1a_1 = \cdots = r_na_n = 0$, and $0 \notin S$.

$\varphi_{A_1''}$. It is clear that φ^* is an isomorphism equal to φ on G_1; these follow from the corresponding properties of the mappings φ_{A_1}. Finally $G_1^{(p)}\varphi^* = G_2^{(p)}$, for $G_2^{(p)}$ is generated by G_2. Thus φ^* has the desired properties. It is easy to see by the uniqueness of the φ_{A_1} s that any extension of φ coincides with φ_{A_1} on $A_1^{(p)}$, and so with φ^* on the whole $G_1^{(p)}$. Therefore φ^* is the only possible extension of φ, as it was stated.

Theorem 7. *A group has a p-adic closure if and only if its torsion subgroup is a p-group.*

PROOF. First let G be a divisible group the torsion subgroup of which is a p-group P. Then G can be decomposed into a direct sum

$$G = P + \sum_{\nu \in \Gamma} G_\nu$$

where the G_ν s $(\nu \in \Gamma)$ are isomorphic to the additive group of rational numbers. We show that the module

$$G^{(p)} = P^{(p)} + \sum_{\nu \in \Gamma} G_\nu^{(p)}$$

is a p-adic closure of G, where the $G_\nu^{(p)}$ s $(\nu \in \Gamma)$ are copies of the field of p-adic numbers, regarded as a p-adic module, and $P^{(p)}$ is a p-adic module having P as its additive group. It is clear that G generates $G^{(p)}$. Let $a_1, \ldots, a_n$ be independent elements of the group G. Each a_j can be expressed as a sum

$$a_j = a_{1j} + \cdots + a_{mj} + a_j' \qquad (j = 1, \ldots, n)$$

where $a_{ij} \in G_{\nu_i}$ $(i = 1, \ldots, m;\ \nu_i \in \Gamma)$ and $a_j' \in P$. We may restrict our considerations to the case when $a_1, \ldots, a_n$ are not contained in P. The independence of the group elements $a_1, \ldots, a_n$ means that the system of equations

$$\text{(1)} \qquad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \\ a_1'x_1 + \cdots + a_n'x_n = 0 \end{cases}$$

has only trivial solution in rational integers. This implies that the system of equations

$$\text{(2)} \qquad \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \cdots\cdots\cdots\cdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{cases}$$

has only trivial solution in rational integers. Indeed, in the contrary case one

could get a non-trivial solution of (1) by multiplying the non-trivial solution of (2) with a suitable power of p. Therefore the matrix

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

has a non-zero determinant of order n. But this implies that (2) and so (1) has only trivial solution in p-adic integers. Thus $a_1, \ldots, a_n$ are independent elements of the module $G^{(p)}$. This proves that $G^{(p)}$ is a p-adic closure of G.

Now let G be an arbitrary group the torsion subgroup of which is a p-group. It is easy to see that G can be embedded in a divisible group $\hat{G}$ the torsion subgroup of which is also a p-group. By the formerly treated special case of our proposition, G has a p-adic closure $\hat{G}^{(p)}$. The submodule of $\hat{G}^{(p)}$ generated by G is a p-adic closure of G.

Conversely it is evident that the torsion subgroup of any subgroup of a p-adic module is a p-group.

This completes the proof of our theorem.

Theorem 8. *Let $H^{(p)}$ be a p-adic closure of the torsion-free group H and P be a p-group. Then* $\mathrm{Ext}(P, H) = 0$ *holds if and only if* $\mathrm{Ext}_p(P, H^{(p)}) = 0$.[12])

Proof. Suppose that $\mathrm{Ext}(P, H) = 0$. Let the module $G^{(p)}$ be an arbitrary extension of the module P by $H^{(p)}$; the elements of $G^{(p)}/P$ are identified with the elements of $H^{(p)}$. Let G be the subgroup of $G^{(p)}$ for which $G/P = H$ holds. Then by our hypothesis P is a direct summand of the group G: $G = P + H'$. We shall prove that $G^{(p)} = P + H'^{(p)}$, where $H'^{(p)}$ is the submodule of $G^{(p)}$ generated by H'.

First we show that $P \cap H'^{(p)} = 0$. Let g be an element common to P and $H'^{(p)}$. There exists a finitely generated subgroup A' of H' in such a way that g is contained in the submodule of $G^{(p)}$ generated by A'. By the fundamental theorem of finitely generated groups, A' has a basis $a_1, \ldots, a_n$. Clearly, $\bar{a}_1, \ldots, \bar{a}_n$ are independent elements of the group G/P (bars indicate cosets modulo P) since $P \cap H' = 0$. Therefore $\bar{a}_1, \ldots, \bar{a}_n$ are independent in the module $G^{(p)}/P$, for the module $G^{(p)}/P$ is a p-adic closure of the group G/P. It follows that $a_1, \ldots, a_n$ are independent in the module $G^{(p)}$. Now, taking into account that $a_1, \ldots, a_n$ are of infinite order and $g \in P$ belongs to the submodule of $G^{(p)}$ generated by $a_1, \ldots, a_n$, we get $g = 0$, i. e. $P \cap H'^{(p)} = 0$.

We have to prove that P and $H'^{(p)}$ generate the module $G^{(p)}$. By the relation $G = P + H'$ it is sufficient to show that G generates the module

[12]) In the latter relation P is considered as a p-adic module.

$G^{(p)}$. Let us take a submodule M of $G^{(p)}$ containing G. Then $M/P = G^{(p)}/P$ since $G/P = H$ generates the module $G^{(p)}/P = H^{(p)}$. Thus indeed $G^{(p)}$ is generated by G and so $G^{(p)} = P + H'^{(p)}$, i. e. $\mathrm{Ext}_p(P, H^{(p)}) = 0$.

Suppose that $\mathrm{Ext}_p(P, H^{(p)}) = 0$. Let the group G be an arbitrary extension of the group P by H. By Theorem 7 there exists a p-adic closure $G^{(p)}$ of G. [13])

First we show that P is the torsion submodule of $G^{(p)}$. Let $P^{(p)}$ be the torsion submodule of $G^{(p)}$. Clearly, $P \subseteq P^{(p)}$. Conversely, let g be an element of finite order of $G^{(p)}$. There exists such a finitely generated subgroup A of G that g is contained in the submodule of $G^{(p)}$ generated by A. By the fundamental theorem of finitely generated groups, A has a basis $a_1, \ldots, a_n$. Then g can be represented in the form

$$g = r_1 a_{i_1} + \cdots + r_m a_{i_m} \qquad (0 < i_1 < \cdots < i_m \leqq n)$$

where, excluding the trivial case $g = 0$, we may suppose that $r_1 \ldots, r_m$ are p-adic integers different from 0. The element g is of finite order, so $a_{i_1}, \ldots, a_{i_m}$ have finite order too. This implies $g \in P$. So indeed $P^{(p)} = P$.

Now we prove that the module $G^{(p)}/P$ is a p-adic closure of the group G/P. The module $G^{(p)}/P$ is generated by G/P. Indeed if M/P contains G/P (M is a submodule of $G^{(p)}$ containing P) then $G \subseteq M$, thus $M = G^{(p)}$. On the other hand, let $\bar{a}_1, \ldots, \bar{a}_n$ be independent elements of the group G/P. It follows that $a_1, \ldots, a_n$ are independent in the group G. Therefore $a_1, \ldots, a_n$ are independent in the module $G^{(p)}$, for the module $G^{(p)}$ is a p-adic closure of the group G. Taking into account that $a_1, \ldots, a_n$ are of infinite order, the independence of $\bar{a}_1, \ldots, \bar{a}_n$ in the module $G^{(p)}/P$ is established. So indeed, $G^{(p)}/P$ is a p-adic closure of G/P.

It follows from this by Theorem 6 that $G^{(p)}/P \cong H^{(p)}$. Therefore by our assumption P is a direct summand of the module $G^{(p)}$ and so of the group G. With this we proved $\mathrm{Ext}(P, H) = 0$.

§ 4. Extensions of groups. The main results.

In this section we are dealing with the splitting problems of mixed groups, making use of the results of the preceding § s. Theorem 9 and Theorem 10 are analogues of Theorem 4 and of Theorem 5; p-adic torsion modules are replaced by p-groups and p-adic torsion-free modules by torsion-free groups. Theorem 11 solves the second problem of § 1 for groups having countable p-adic dimension for at least one p. Finally we give answer to a question of R. BAER.

[13]) The letters used in the second part of the proof have naturally another meaning than before.

Definition. *By the p-adic dimension of a group we mean the dimension of its p-adic closure (if it exists).*

Theorem 9. *Let H be a torsion-free group of countable p-adic dimension, and P a p-group. Then* $\mathrm{Ext}(P, H) = 0$ *holds if and only if at least one of the following two conditions is satisfied.*

(1) *P is a direct sum of a group of bounded order and a divisible group.*

(2) *The p-adic closure of H is a p-adic free module.*

The proof is clear by Theorem 4 and Theorem 8.

Theorem 10. *Let H be a torsion-free group of countable p-adic dimension. Then* $\mathrm{Ext}(P, H) = 0$ *holds with every p-group P if and only if the p-adic closure of H is a p-adic free module.*

The proof is clear by Theorem 9.

Theorem 11. *Let H be a torsion-free group having countable p-adic dimension for at least one p. Then* $\mathrm{Ext}(T, H) = 0$ *holds with every torsion group T if and only if H is a free group.*

PROOF. Suppose that $\mathrm{Ext}(T, H) = 0$ holds with every torsion group T. Then by Theorem 10 the p-adic closure $H^{(p)}$ of the group H is a free module. Since $H^{(p)}$ is a module of countable dimension, H is a countable group. Making use of the assumption $\mathrm{Ext}(T, H) = 0$, we conclude that H is a free group (see R. BAER [1]).

Conversely, if H is a free group, then, as it is well known, the relation $\mathrm{Ext}(T, H) = 0$ is valid for any group T.

This completes the proof.

Solution of a problem raised by R. BAER.

The question under consideration is the following: *does the relation* $\mathrm{Ext}(T, H) = 0$ *hold with every torsion group T if H is the group of all sequences of rational integers?* (R. BAER [1]). Our answer will be *negative* even in the case when $T = \sum_n C(p^n)$.

First of all the following statement is an immediate corollary of Theorem 4, Theorem 8 and Lemma 2.

Let H be a torsion-free group and P a p-group. If $\mathrm{Ext}(P, H) = 0$ *holds then either P is a direct sum of a group of bounded order and a divisible group, or every submodule of countable dimension of the p-adic closure of H is a free module.*

So it is sufficient to prove the existence of a subgroup of the group of all sequences of rational integers the p-adic closure of which is not a free module and its dimension is countable.

Let G_p be the group of all sequences $r_1, \ldots, r_n, \ldots$ of rational integers which have the following property: the sequence of p-heights of $r_1, \ldots, r_n, \ldots$ tends to infinity. Let G_0 be the group of all sequences of rational integers the elements of which are all 0 at most with a finite number of exceptions. Let $G_p^{(p)}$ be a p-adic closure of G_p and $G_0^{(p)}$ the submodule of $G_p^{(p)}$ generated by the subgroup $G_0 \subseteqq G_p$.

First we prove that $G_0^{(p)}$ is a basic submodule of $G_p^{(p)}$. It is obvious that $G_0^{(p)}$ is a free module, for $G_0^{(p)}$ is a p-adic closure of G_0. We show that $G_0^{(p)}$ is a pure submodule of $G_p^{(p)}$. Suppose that for an element $g \in G_p^{(p)}$ a relation $p^k g \in G_0^{(p)}$ holds. G_p has a finitely generated subgroup A in such a way that the submodule of $G_p^{(p)}$ generated by A contains g and the submodule generated by $G_0 \cap A$ contains $p^k g$. It is clear that G_p^*/G_0 is a torsion-free group, thus $G_0 \cap A$ is a direct summand of A. Therefore by the fundamental theorem of finitely generated groups, A has a basis $a_1, \ldots, a_n$ some members of which generate $G_0 \cap A$. From the representation

$$g = r_1 a_1 + \cdots + r_n a_n$$

where $r_1, \ldots, r_n$ are p-adic integers, we obtain

$$p^k g = p^k r_1 a_1 + \cdots + p^k r_n a_n.$$

In the latter sum the coefficients of those a_i s which do not belong to G_0 are 0 s since $a_1, \ldots, a_n$ are independent in the module $G_p^{(p)}$ too. This implies

$$g = r_1 a_1 + \cdots + r_n a_n \in G_0^{(p)},$$

i. e. $G_0^{(p)}$ is a pure submodule of $G_p^{(p)}$. Finally we have to show that $G_p^{(p)}/G_0^{(p)}$ is divisible. In order to prove this let us consider an arbitrary equation $p\bar{x} = \bar{g}$ ($g \in G_p^{(p)}$; bars indicate cosets modulo $G_0^{(p)}$). The element g can be expressed as a sum

$$g = r_1 a_1 + \cdots + r_n a_n \qquad (a_1, \ldots, a_n \in G_p)$$

where $r_1, \ldots, r_n$ are p-adic integers. It follows by the construction of G_p and G_0 that $p(G_p/G_0) = G_p/G_0$; thus there exist elements $x_i \in G_0$ ($i = 1, \ldots, n$) satisfying

$$p x_i - a_i \in G_0 \subseteqq G_0^{(p)},$$

i. e. $p\bar{x}_i = \bar{a}_i$. Therefore $\overline{r_1 x_1 + \cdots + r_n x_n}$ is the required solution of $p\bar{x} = \bar{g}$. So $G_p^{(p)}/G_0^{(p)}$ is a divisible module, and this proves that $G_0^{(p)}$ is a basic submodule of $G_p^{(p)}$.

Clearly, $G_0^{(p)}$ is a module of countable rank, thus, by the former part of the proof, $G_p^{(p)}$ is of countable dimension. On the other hand, it can be easily seen that the rank of the group G_p and so that of the module $G_p^{(p)}$ is not countable. Therefore $G_p^{(p)}$ is not a free module. This proves by the preceding remark that $\operatorname{Ext}(T, H) \neq 0$ even in the case when $T = \sum_n C(p^n)$.

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