

On convex solutions of the functional equation

$$g[\alpha(x)] - g(x) = \varphi(x).$$

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1. In the present paper I want to show the uniqueness of convex solutions of the functional equation

$$(1) \quad g[\alpha(x)] - g(x) = \varphi(x), \quad x \in [a, \infty),$$

where $g(x)$ denotes the required function, and $\alpha(x)$ and $\varphi(x)$ denote known functions, and I shall not treat the problem of the existence of such solutions.

It is easy to be verified (see e. g. [3]) that the equation (1) possesses (with the continuous functions $\alpha(x)$ and $\varphi(x)$) infinitely many continuous solutions. Their number does not even diminish essentially if we require the regularity of the solutions.¹⁾ However, if we require the solution to be convex, then it turns out that with some hypotheses on the functions $\alpha(x)$ and $\varphi(x)$ there may exist at most one such solution up to an additive constant.

This fact has been known formerly for the equation

$$(2) \quad g(x+1) - g(x) = \log x \quad x > 0.$$

The unique convex solution of the equation (2) which fulfils the condition $g(0) = 0$, is the function $g(x) = \log \Gamma(x)$. Thus the equation (2) may be used for defining Euler's Γ function (see e. g. [1]).

This result was generalized for the equation

$$g(x+1) - g(x) = \varphi(x)$$

by KRULL [4] and, independently, by the author of this paper [5]. The present note is a direct generalization of the above results.

¹⁾ If, for example, $g(x)$ is an analytic solution of the equation

$$g(x+h) - g(x) = \varphi(x),$$

then the functions

$$f(x) = g(x) + A \sin \left(\frac{2\pi}{h} x + B \right) + C,$$

where A, B, C are arbitrary constants, are also analytic solutions of this equation.

2. We call the function $f(x)$ convex in an interval (a, b) if for every x_1, x_2 , belonging to (a, b) and for every $0 < \lambda < 1$, the inequality

$$f[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

holds.²⁾

We call the function $f(x)$ concave if it satisfies the inequality:

$$f[\lambda x_1 + (1 - \lambda)x_2] \geq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

Lemma 1. *If the function $f(x)$ is convex (concave) in an interval (a, b) then for each $x_0 \in (a, b)$ the functions*

$$f_1(x) = \frac{f(x_0 + x) - f(x_0)}{x}$$

$$f_2(x) = \frac{f(x + x_0) - f(x)}{x_0}$$

are increasing (decreasing) in the interval (a, b) .

The proof of this lemma is to be found in [2].

We denote by $\alpha^n(x)$ the n -th iteration of the function $\alpha(x)$, i.e. we put:

$$\alpha^0(x) = x$$

$$\alpha^{n+1}(x) = \alpha[\alpha^n(x)].$$

Lemma 2. *If the function $\alpha(x)$ is continuous and strictly increasing in $[a, \infty)$, and $\alpha(x) > x$ in $[a, \infty)$, then for every $x \in [a, \infty)$ the sequence $\{\alpha^n(x)\}$ is strictly increasing and*

$$\lim_{n \rightarrow \infty} \alpha^n(x) = \infty.$$

The proof of this lemma may be found e. g. in [6].

Lemma 3. *If the function $\alpha(x)$ is concave, continuous and strictly increasing in $[a, \infty)$ and $\alpha(x) > x$ in $[a, \infty)$, then for every $x_1, x_2 \in [a, \infty)$*

$$\frac{\alpha(x_1) - \alpha(x_2)}{x_1 - x_2} \geq 1.$$

PROOF. On account of Lemma 1 the function $\frac{\alpha(x+a) - \alpha(a)}{x}$ is decreasing, thus there exists the limit

$$\lim_{x \rightarrow \infty} \frac{\alpha(x+a) - \alpha(a)}{x} = \alpha_0.$$

²⁾ Convex functions are often defined by the inequality:

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}.$$

These definitions are equivalent under the hypothesis of the continuity of the function $f(x)$. It may be proved that the convex and measurable function is continuous ([7], [2]).

Consequently there exists also the limit

$$\lim_{x \rightarrow \infty} \frac{\alpha(x)}{x} = \lim_{x \rightarrow \infty} \frac{\alpha(a+x)}{a+x} = \lim_{x \rightarrow \infty} \left[\frac{\alpha(x+a) - \alpha(a)}{x} \cdot \frac{x}{x+a} + \frac{\alpha(a)}{x+a} \right] = \alpha_0,$$

and according to the inequality $\alpha(x) > x$, we have $\alpha_0 \geq 1$.

In a similar manner we have by Lemma 1:

$$\frac{\alpha(x_1) - \alpha(x_2)}{x_1 - x_2} \geq \lim_{x_1 \rightarrow \infty} \frac{\alpha(x_1) - \alpha(x_2)}{x_1 - x_2} = \lim_{x_1 \rightarrow \infty} \frac{\alpha(x_1)}{x_1 - x_2} = \lim_{x_1 \rightarrow \infty} \frac{\frac{\alpha(x_1)}{x_1}}{1 - \frac{x_2}{x_1}} = \alpha_0 \geq 1,$$

what was to be proved.

Let us denote, for fixed $x \in (a, \alpha(a))$: $x_n = \alpha^n(x)$, $a_n = \alpha^n(a)$,

$\delta_n(x) = x_n - a_n$, $\Delta_n = a_n - a_{n-1}$, $I_n(x) = \frac{\delta_n(x)}{\Delta_n}$. The immediate consequence of Lemma 3 is

Lemma 4. *If the function $\alpha(x)$ fulfils the hypotheses of Lemma 3, then the sequence $\{\Delta_n\}$ is increasing.*

Now we shall show the following

Lemma 5. *If the function $\alpha(x)$ fulfils the hypotheses of Lemma 3, then for fixed x the sequence $\{I_n(x)\}$ is decreasing.*

PROOF We have by Lemma 1:

$$\frac{x_{n+1} - a_{n+1}}{x_n - a_n} = \frac{\alpha(x_n) - \alpha(a_n)}{x_n - a_n} \leq \frac{\alpha(x_n) - \alpha(a_{n-1})}{x_n - a_{n-1}} \leq \frac{\alpha(a_n) - \alpha(a_{n-1})}{a_n - a_{n-1}} = \frac{a_{n+1} - a_n}{a_n - a_{n-1}}$$

Hence

$$I_{n+1}(x) = \frac{x_{n+1} - a_{n+1}}{a_{n+1} - a_n} \leq \frac{x_n - a_n}{a_n - a_{n-1}} = I_n(x), \text{ what completes the proof.}$$

For further purposes we shall need also the following:

Lemma 6. *If the function $\alpha(x)$ fulfils the hypotheses of Lemma 3 and if moreover $\lim_{x \rightarrow \infty} \frac{\alpha(x)}{x} = \alpha_0 > 1$, then there exist a number $\delta > 0$ and an index N such that for $n > N$:*

$$a_{n+1} + a_{n-1} - 2a_n > \delta a_n.$$

PROOF. We have

$$\frac{\alpha[\alpha(x)] + x}{2\alpha(x)} = \frac{\frac{\alpha[\alpha(x)]}{\alpha(x)} \cdot \frac{\alpha(x)}{x} + 1}{2 \frac{\alpha(x)}{x}} \xrightarrow{x \rightarrow \infty} \frac{\alpha_0^2 + 1}{2\alpha_0} > 1.$$

Consequently, we can choose the numbers $\delta > 0$ and $A > 0$ such that for $x > A$

$$\frac{\alpha[\alpha(x)] + x}{2\alpha(x)} > 1 + \frac{\delta}{2}.$$

Hence for $x > A$:

$$(3) \quad \alpha[\alpha(x)] + x - 2\alpha(x) > \delta\alpha(x).$$

Since $\lim_{n \rightarrow \infty} a_n = \infty$, we can choose the index N such that for $n > N$ $a_{n-1} > A$.

Putting in (3) $x = a_{n-1}$ we obtain (for $n > N$):

$$a_{n+1} + a_{n-1} - 2a_n > \delta a_n,$$

what was to be proved.

3. In what follows we shall assume that the function $\alpha(x)$ is concave, continuous in an interval $[a, \infty)$ and $\alpha(x) > x$ in $[a, \infty)$. Hence it follows that $\alpha(x)$ is strictly increasing in $[a, \infty)$. Moreover we shall suppose that the function $\varphi(x)$ is continuous in $[a, \infty)$, and that it is positive and increasing for sufficiently great x , and that

$$(4) \quad \lim_{n \rightarrow \infty} [\varphi(a_n) - \varphi(a_{n-1})] = 0.$$

Let us suppose that the function $g(x)$ satisfies the equation (1) and that $g(x)$ is convex in $[a, \infty)$. According to Lemma 1, for arbitrary $x \in (a, \alpha(a))$ hold the inequalities (using the notation from the preceding sections):

$$(5) \quad \frac{g(a_n) - g(a_{n-1})}{\Delta_n} \leq \frac{g(x_n) - g(a_n)}{\delta_n(x)}$$

$$(6) \quad \frac{g(x_n) - g(a_n)}{\delta_n(x)} \leq \frac{g(a_{n+1}) - g(a_n)}{\Delta_{n+1}}$$

We have from (5) and (1):

$$(7) \quad \delta_n(x) \varphi(a_{n-1}) \leq \Delta_n [g(x_n) - g(a_n)].$$

We have from (6) and (1):

$$(8) \quad \Delta_{n+1} [g(x_n) - g(a_n)] \leq \delta_n(x) \varphi(a_n).$$

Next:

$$\begin{aligned} g(x_n) - g(a_n) &= g(x_n) - g(x) + g(x) - g(a_n) + g(a) - g(a) = \\ &= \sum_{k=0}^{n-1} [g(x_{k+1}) - g(x_k)] + g(x) - \sum_{k=0}^{n-1} [g(a_{k+1}) - g(a_k)] - g(a) = \\ &= \sum_{k=0}^{n-1} \varphi(x_k) + g(x) - g(a) - \sum_{k=0}^{n-1} \varphi(a_k) = g(x) - g(a) + \sum_{k=0}^{n-1} [\varphi(x_k) - \varphi(a_k)]. \end{aligned}$$

Hence, according to (7)

$$\frac{\delta_n(x)}{\Delta_n} \varphi(a_{n-1}) \leq g(x) - g(a) + \sum_{k=0}^{n-1} [\varphi(x_k) - \varphi(a_k)],$$

that is

$$(9) \quad I_n(x)\varphi(a_{n-1}) + g(a) - \sum_{k=0}^{n-1} [\varphi(x_k) - \varphi(a_k)] \leq g(x).$$

On the other hand we have by (8):

$$g(x) - g(a) + \sum_{k=0}^{n-1} [\varphi(x_k) - \varphi(a_k)] \leq \frac{\delta_n(x)}{\Delta_{n+1}} \varphi(a_n),$$

that is

$$(10) \quad g(x) \leq g(a) - \sum_{k=0}^{n-1} [\varphi(x_k) - \varphi(a_k)] + \frac{\delta_n(x)}{\Delta_{n+1}} \varphi(a_n) + I_n(x)\varphi(a_{n-1}) - I_n(x)\varphi(a_{n-1})$$

Denoting:

$$g_n(x) \stackrel{\text{def}}{=} g(a) + I_n(x)\varphi(a_{n-1}) - \sum_{k=0}^{n-1} [\varphi(x_k) - \varphi(a_k)],$$

we have from (9) and (10)

$$g_n(x) \leq g(x) \leq g_n(x) + I_n(x) \left[\frac{\Delta_n}{\Delta_{n+1}} \varphi(a_n) - \varphi(a_{n-1}) \right].$$

According to Lemma 4 $\frac{\Delta_n}{\Delta_{n+1}} \leq 1$. Since $a_n \xrightarrow{n \rightarrow \infty} \infty$ and $\varphi(x) > 0$ for great x , $\varphi(a_n) > 0$ for great n and

$$g_n(x) \leq g(x) \leq g_n(x) + I_n(x) [\varphi(a_n) - \varphi(a_{n-1})].$$

According to Lemma 5 the sequence $I_n(x)$ is decreasing, and thus bounded. Hence, according to (4)

$$\lim_{n \rightarrow \infty} g_n(x) = g(x) \quad \text{for } x \in (a, \alpha(a)).$$

Since $g_n(a) = g(a)$ for every n ,

$$\lim_{n \rightarrow \infty} g_n(x) = g(x) \quad \text{for } x \in [a, \alpha(a)).$$

Thus every convex solution of the equation (1), which assumes the value $g(a)$ at the point $x = a$, must be in the interval $[a, \alpha(a))$ the limit of the sequence $g_n(x)$. Since the function $g(x)$, satisfying (1), is unambiguously determined by its values from the interval $[a, \alpha(a))$, (see e. g. [3]), hence follows the uniqueness of the convex solutions of the equation (1).

REMARK. In the above considerations from the hypotheses on the function $\varphi(x)$ we used only positiveness of $\varphi(x)$ for great x and the condition (4). Nevertheless, we shall show that if the equation (1) possesses a convex solution $g(x)$, then the function $\varphi(x) (=g[\alpha(x)] - g(x))$ being positive must be increasing for great x .

From the condition $\varphi(x) > 0$ for great x follows that the function $g(x)$ is increasing for $x > A$. Let us take arbitrary $x_1 > x_2 > A$. According to Lemma 3

$$\frac{\alpha(x_1) - \alpha(x_2)}{x_1 - x_2} \geq 1,$$

that is

$$(11) \quad \alpha(x_1) \geq x_1 + \alpha(x_2) - x_2.$$

Denoting $h = \alpha(x_2) - x_2$, we have by Lemma 1:

$$\frac{g(x_2 + h) - g(x_2)}{h} \leq \frac{g(x_1 + h) - g(x_1)}{h},$$

whence

$$g(x_2 + h) - g(x_2) \leq g(x_1 + h) - g(x_1),$$

that is

$$g[\alpha(x_2)] - g(x_2) \leq g(x_1 + \alpha(x_2) - x_2) - g(x_1).$$

Since $g(x)$ is increasing, we have from above, according to (11)

$$g[\alpha(x_2)] - g(x_2) \leq g[\alpha(x_1)] - g(x_1),$$

that is

$$\varphi(x_2) \leq \varphi(x_1).$$

4. In the preceding section we established the uniqueness of the convex solutions of the equation (1), which assume a given value at the point $x = a$. We shall show, however, that if $\lim_{x \rightarrow \infty} \frac{\alpha(x)}{x} > 1$, then the equation (1) with the function $\varphi(x)$ fulfilling the hypotheses from the preceding section, has no convex solutions at all. Namely, we shall show that if $\lim_{x \rightarrow \infty} \frac{\alpha(x)}{x} = \alpha_0 > 1$ and if $g(x)$ is a convex solution of the equation (1), then

$$(12) \quad \lim_{n \rightarrow \infty} [\varphi(a_n) - \varphi(a_{n-1})] = \lim_{n \rightarrow \infty} [g(a_{n+1}) + g(a_{n-1}) - 2g(a_n)] = \infty.$$

The function $g(x)$, being convex, satisfies for every $x_1, x_2 \in [a, \infty)$ and for every $0 < \lambda < 1$ the inequality:

$$g[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda g(x_1) + (1 - \lambda)g(x_2).$$

Let us put $x_1 = a_n$, $x_2 = a_{n+1}$, $\lambda = \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}}$. Evidently $0 < \lambda < 1$.

We have:

$$\begin{aligned} \lambda x_1 + (1 - \lambda)x_2 &= \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}} a_{n-1} + \frac{(a_{n+1} - a_{n-1}) - (a_{n+1} - a_n)}{a_{n+1} - a_{n-1}} a_{n+1} = \\ &= \frac{(a_{n+1} - a_n) a_{n-1} + (a_n - a_{n-1}) a_{n+1}}{a_{n+1} - a_{n-1}} = a_n. \end{aligned}$$

Consequently:

$$(13) \quad g(a_n) \leq g(a_{n+1}) + \lambda[g(a_{n-1}) - g(a_{n+1})].$$

Let us denote

$$m_n = \frac{g(a_{n+1}) - g(a_{n-1})}{a_{n+1} - a_{n-1}}.$$

We have

$$g(a_{n+1}) = m_n(a_{n+1} - a_{n-1}) + g(a_{n-1}).$$

According to (13)

$$\begin{aligned} g(a_n) &\leq g(a_{n+1}) + \frac{a_{n+1} - a_n}{a_{n+1} - a_{n-1}} [g(a_{n-1}) - g(a_{n+1})] = g(a_{n+1}) - m_n(a_{n+1} - a_n) = \\ &= g(a_{n+1}) - m_n(a_{n+1} - a_n + a_{n-1} - a_{n-1}) = g(a_{n+1}) + m_n(a_n - a_{n-1}) - g(a_{n+1}) + \\ &\quad + g(a_{n-1}) = g(a_{n-1}) + m_n(a_n - a_{n-1}). \end{aligned}$$

Hence

$$g(a_{n+1}) + g(a_{n-1}) - 2g(a_n) \geq m_n(a_{n+1} + a_{n-1} - 2a_n).$$

The function $g(x)$ is increasing for great x , thus there exists an index $N_1 > N$ (where N denotes the index occurring in Lemma 6) such that $m_{N_1} > 0$. Further, on account of Lemma 1, $m_n > m_{N_1}$ for $n > N_1$. Hence by Lemma 6, for $n > N_1$:

$$g(a_{n+1}) + g(a_{n-1}) - 2g(a_n) \geq m_{N_1} \delta a_n,$$

whence the relation (12) follows immediately.

Thus we have proved the following

Theorem. *If the function $\alpha(x)$ is concave and continuous in an interval $[a, \infty)$, moreover $\alpha(x) > x$ in $[a, \infty)$, and if the function $\varphi(x)$ is continuous in $[a, \infty)$, positive (and then also increasing) for sufficiently great x , and fulfils the condition (4), then there exists at most one convex function $g(x)$, which satisfies the equation (1) in the interval $[a, \infty)$ and assumes a given value for $x = a$. The necessary condition of existence of such a solution is the relation*

$$\lim_{x \rightarrow \infty} \frac{\alpha(x)}{x} = 1.$$

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