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On the local compactness and spaces of subsets

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Abstract. In the present paper the local compactness of the union of compact sets, and some relationships between the local compactness property of a topological space X and the space of its compact subsets are studied. Finally an application in the theory of multifunctions is given.

1. Notations and premilinares

Let $(X,\mathcal{T})$ be a topological space. Then

$$\begin{aligned}\mathcal{P}_0(X)&=\{E\subset X \mid \text{E is not empty}\}\\ \mathcal{K}(X)&=\{E\in\mathcal{P}_0(X)\mid \text{E is compact}\}\\ \mathcal{F}_1(X)&=\{\{x\}\mid x\in X\}\end{aligned}$$

If $\{A_i\}_{i\in I}$ is a family from $\mathcal{P}_0(X)$, then

$$\begin{aligned}\langle\{A_i\}_{i\in I}\rangle&=\{E\in\mathcal{P}_0(X)\mid E\subset\bigcup_{i\in I}A_i,\ E\cap A_i\neq\emptyset,\ i\in I\}\\ [\{A_i\}_{i\in I}]&=\{E\in\mathcal{P}_0(X)\mid E\cap A_i\neq\emptyset,\ i\in I\}.\end{aligned}$$

Definition 1. The topology $\mathcal{T}_u^{\uparrow}$ ($\mathcal{T}_l^{\uparrow}$; $\mathcal{T}^{\uparrow}$) generated by the base (subbase)

$$\begin{gathered}\mathcal{S}_u=\{\langle\{G\}\rangle\mid G\in\mathcal{T}\}\\ (\mathcal{S}_l=\{[\{G\}]\mid G\in\mathcal{T}\}\ ;\ \mathcal{S}_u\cup\mathcal{S}_l)\end{gathered}$$

is called upper semi-finite (lower semi-finite; finite) topology on $\mathcal{P}_0(X)$.

2. Local compactness in X and $\mathcal{K}(X)$

Theorem 1. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{A}$ be a family of compact subsets X. If $\mathcal{A}$ is open and locally compact in the space $(\mathcal{K}(X), \mathcal{T}_u^{\uparrow})$, then the union of members of family $\mathcal{A}$ is a locally compact in $(X, \mathcal{T})$.*

Proof. Since $\mathcal{A}$ is locally compact in $(\mathcal{K}(X), \mathcal{T}_u^{\uparrow})$, it results that for each $A \in \mathcal{A}$ there exists a compact neighbourhood $\mathcal{V} \subset \mathcal{A}$. Hence there exists an open subset G of X such that $\langle\{G\}\rangle \cap \mathcal{A} \subset \mathcal{V}$. Therefore if $\mathcal{A}$ locally compact, then there holds the assertion:

(i) For each $A \in \mathcal{A}$ there exists a compact subset $\mathcal{V} \subset \mathcal{A}$ with $A \in \mathcal{A}$, and there is an open subset G of X, $A \subset G$, such that $\bigcup\limits_{L \in \langle\{G\}\rangle \cap \mathcal{A}} L \subset \bigcup\limits_{V \in \mathcal{V}} V$. Since $\mathcal{A}$ is open in the topology $\mathcal{T}_u^{\uparrow}$ relativized to $\mathcal{K}(X)$, it results that there exists an open subset H of X such that $A \in \langle\{H\}\rangle \cap \mathcal{K}(X) \subset \mathcal{A}$, for each $A \in \mathcal{A}$. Then $G_0 = \bigcup\limits_{L \in \langle\{H\}\rangle \cap \mathcal{K}(X)} L$ is open in the subspace $B = \bigcup\limits_{A \in \mathcal{A}} A \subset X$. Therefore, the fact that the family $\mathcal{A}$ is open it implies the assertion:

(ii) For each $A \in \mathcal{A}$ and for any open subset H of X, which contains A, there exists an open subset G_0 of H such that $A \in \bigcup\limits_{L \subset \langle\{G0\}\rangle \cap \mathcal{A}} L = G_0 \cap B$.

From the statements (i) and (ii) it results that B is a locally compact subspace of $(X, \mathcal{T})$.

Lemma 1. *If $(X, \mathcal{T})$ is compact topological space and $2^X \subset Q(X) \subset \mathcal{P}_0(X)$, then $Q(X)$ is compact in the topology $\mathcal{T}^{\uparrow}$.*

Proof. It will be prove that $Q(X)$ is compact by using Alexander's theorem. Hence, let there be a cover of $Q(X)$ by elements $\mathcal{S}_u \cap \mathcal{S}_l$, that is

$$Q(X) \subset (\bigcup_{j \in J} \langle\{G_j\}\rangle) \cup (\bigcup_{l \in L} [\{H_l\}]),$$

where $G_j, H_l \in \mathcal{T}$; $j \in J$, $l \in L$.

Let $H = \bigcup\limits_{l \in L} H_l$ and let $K = X \backslash H$. If $H = X$ then there exists a finite subcover of the cover $\bigcup\limits_{l \in L} H_l$ of X. Hence $X = \bigcup\limits_{k=1}^{n} H_{l_k}$. Then it results

$$Q(X) \subset \mathcal{P}_0(X) = [\{X\}] = [\{\bigcup_{k=1}^{n} H_{l_k}\}] = \bigcup_{k=1}^{n} [\{H_{l_k}\}]$$

that is $Q(X)$ is compact.

If $H \neq X$, then $K \in Q(X)$, and $K \notin [\{H_l\}]$ for each $l \in L$. Hence, there is $j_0 \in J$ such that $K \in \langle\{G_{j_0}\}\rangle$, that is $K \subset G_{j_0}$. Let $M = X \backslash G_{j_0}$. Since M is closed and X is compact, it follows that M is compact. From $M \subset H$ it results that there is a finite subcover of M with sets H_l, that is $M \subset \bigcup_{i=1}^{m} H_{l_i}$.

We will prove that

$$Q(X) \subset \langle\{G_j\}\rangle \cup (\bigcup_{i=1}^{m} [\{H_{l_i}\}]).$$

Let $A \in Q(X)$. If $A \cap M \neq \emptyset$, then there is $i_0 \in \overline{1,m}$ such that $A \cap H_{l_{i_0}} \neq \emptyset$, that is $A \in [\{H_{l_{i_0}}\}]$.
If $A \cap M = \emptyset$, then $A \subset G_{j_0}$, whence $A \in \langle\{G_{j_0}\}\rangle$, and the lemma is proved.

Theorem 2. *If $(X, \mathcal{T})$ is locally compact topological space, then the topological space $(\mathcal{K}(X), \mathcal{T}^{\uparrow})$ is locally compact.*

Proof. Let $A \in \mathcal{K}(X)$. For $x \in A$, there is a compact neighbourhood $V_x \subset X$. Then, it results $A \subset \bigcup_{x \in A} G_x \subset \bigcup_{x \in A} V_x$, where $x \in G_x \in \mathcal{T}$, and $G_x \subset V_x$. Since $A \in \mathcal{K}(X)$, it follows that there exist $x_1, x_2, \ldots, x_n \in A$ such that $A \subset \bigcup_{i=1}^{n} G_{x_i} \subset \bigcup_{i=1}^{n} V_{x_i} = V \in \mathcal{K}(X)$. From here it result that $A \in \langle\{\bigcup_{i=1}^{n} G_{x_i}\}\rangle \subset \langle\{V\}\rangle$, that is $\langle\{V\}\rangle$ is a neighbourhood of A in the topology $\mathcal{T}^{\uparrow}$. According to the previous lemma, it follows that $\langle\{V\}\rangle \cap \mathcal{K}(X)$ is compact in $\mathcal{T}^{\uparrow}$ relativized to $\mathcal{K}(X)$. Thus the theorem is proved.

Remark. The previous theorem holds if instead $\mathcal{T}^{\uparrow}$ we take $\mathcal{T}_u^{\uparrow}$.

Lemma 2. *Let $(X, \mathcal{T})$ be a topological space. If $\mathcal{F}_1(X) \subset \mathcal{R}(X) \subset \mathcal{P}_0(X)$ and $(\mathcal{R}(X), \mathcal{T}_l^{\uparrow})$ is compact, then $(X, \mathcal{T})$ is compact.*

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of X. Then $[\{X\}] = [\{\bigcup_{i \in I} G_i\}] = \bigcup_{i \in I} [\{G_i\}]$, that is $\mathcal{P}_0(X) = \bigcup_{i \in I} [\{G_i\}]$. Since $\mathcal{R}(X) \subset \mathcal{P}_0(X)$ is compact in the topology $\mathcal{T}_l^{\uparrow}$, it results that there is a finite subcover of cover $\{[\{G_i\}]\}_{i \in I}$ of $\mathcal{R}(X)$. Hence there exist $i_1, i_2, \ldots, i_p \in I$ such that $\mathcal{R}(X) \subset \bigcup_{s=1}^{p} [\{G_{i_s}\}]$. Because $\mathcal{F}_1(X) \subset \mathcal{R}(X)$, it results $X = \bigcup_{s=1}^{p} G_{i_s}$, which is what we set out to prove.

Theorem 3. *If $(X,\mathcal{T})$ is a regular topological space and $\mathcal{F}_1(X) \subset Q(X) \subset \mathcal{K}(X)$ such that $(Q(X),\mathcal{T}^{\uparrow})$ is locally compact, then $(X,\mathcal{T})$ is locally compact.*

Proof. Let $x \in X$, and let $\mathcal{V}_{\{x\}}$ be a compact neigbourhood of $\{x\}$ in the topological space $(Q(X),\mathcal{T}^{\uparrow})$. Then, there exist $G_1, G_2, \dots, G_n \in \mathcal{T}$ such that $\{x\} \in \langle\{G_i\}_{i\in\overline{1,n}}\rangle \cap Q(X) \subset \mathcal{V}_{\{x\}}$. Hence $x \in \bigcap_{i=1}^{n} G_i = G$. Since $(X,\mathcal{T})$ is regular, it results that there exists $W \in \mathcal{T}$ such that $x \in W \subset \overline{W} \subset G$. Then $\{x\} \in \langle\{W\}\rangle \cap Q(X) \subset \langle\{\overline{W}\}\rangle \cap Q(X) \subset \mathcal{V}_{\{x\}}$. Since $\langle\{\overline{W}\}\rangle \cap Q(X)$ is closed in $Q(X)$ endowed with the topology $\mathcal{T}^{\uparrow}$ relativized, it results that it is compact. By the Lemma 2. it result that $\overline{W}$ is compact in $(X,\mathcal{T})$. Therefore $(X,\mathcal{T})$ is locally compact, thus the theorem is proved.

Application. By a multifunction we mean a correspondence denoted $F\colon X \multimap Y$ on a set X into a set Y such that $F(x)$ is a nonempty subset of Y for each $x \in X$.

To a multifunction $F\colon X \multimap Y$ we attach a function $\mathbf{F} : X \to \mathcal{P}_0(Y)$, $\mathbf{F}(x) = F(x)$. We review that a multifunction $F\colon (X,\mathcal{T}_1) \multimap (Y,\mathcal{T}_2)$ is upper semi-continuous iff the corresponding function $\mathbf{F}\colon (X,\mathcal{T}_1) \to (Y,\mathcal{T}_{2u}^{\uparrow})$ is continuous. The multifunction $F\colon (X,\mathcal{T}_1) \to (Y,\mathcal{T}_2)$ is said to be open iff the set $F(A) = \bigcup_{x\in A} F(x)$ is open in Y for all open sets $A \subset X$, and it is called point compact if $F(x)$ is compact subset of Y for all $x \in X$. In the sequel we will use the well-known result [1], that a union of compact family in $\mathcal{T}_{2u}^{\uparrow}$ of compact subsets of Y is a compact set in $(Y,\mathcal{T}_2)$.

Proposition. *Let $F\colon (X,\mathcal{T}_1) \multimap (Y,\mathcal{T}_2)$ be an open upper semi-continuous multifunction onto Y. If $(X,\mathcal{T}_1)$ is locally compact then so $(Y,\mathcal{T}_2)$ does.*

Proof. The family $\mathcal{A} = \{F(x) \mid x \in X\}$ from Y satisfies the conditions (i) and (ii) form the proof of the theorem 1.

Indeed, let $A \in \mathcal{A}$ and $x \in X$ such that $A = F(x)$. Since X is a locally compact, it results that there exists a compact neighbourhood V of x. Let $\mathcal{V} = \mathbf{F}(V)$. $\mathcal{V}$ is compact in $(\mathcal{K}(Y),\mathcal{T}_{2u}^{\uparrow})$ because $\mathbf{F}$ is continuous. Since V is a neighbourhood of x, it results there exists $H \in \mathcal{T}_1$ such that $x \in H \subset V$.

Let $G = F(H) \subset F(V) = \bigcup_{x\in V} F(x) = \bigcup_{A\in\mathbf{F}(V)} A = \bigcup_{A\in\mathcal{V}} A$. It is evident that $\bigcup_{L\in\langle\{G\}\rangle\cap\mathcal{A}} L \subset \bigcup_{A\in\mathcal{V}} A$. Hence (i) holds.

To verify (ii) let $A \in \mathcal{A}$ and let $G \in \mathcal{T}_2$ such that $A \subset G$. Then there exists $x \in X$ with $F(x) = A = \mathbf{F}(x) \in \langle\{G\}\rangle \cap \mathcal{A}$. Since F is upper semi-continuous it results that there exists $H \in \mathcal{T}_1$ such that $F(H) \subset G$.

The set $G_0 = F(H)$ is open and it satisfies the relation $\bigcup\limits_{L \in \langle\{G_0\}\rangle \cap \mathcal{A}} L \subset G_0$, that is the condition (ii) holds.

Taking into account the Theorem 1, it results $F(X) = Y$ is locally compact.

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