A remark on scalar valued multiplicative functions of matrices.

To Professor Otto Varga on his 50th birthday.

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Let $K_n^{\times}$ denote the multiplicative semigroup of n-rowed square matrices over the real (or complex) field K. The mapping $\mathbf{A} \to \varphi\mathbf{A}$ of $K_n^{\times}$ into K is called *multiplicative,* if the equation

$$\varphi(\mathbf{A}\mathbf{B}) = \varphi\mathbf{A}\varphi\mathbf{B}, \qquad \mathbf{A}, \mathbf{B} \in K_n^{\times}$$

holds. M. KUCHARZEWSKI [5] has proved that every mapping $\mathbf{A} \to \varphi\mathbf{A}$ of this form is a multiplicative function (in the usual sense) of det $\mathbf{A}$. M. KUCZMA [6] has simplified the proof of M. KUCHARZEWSKI's theorem. The object of the present paper is to prove this theorem in another way.

We shall use the well known theorem [4] that every matrix $\mathbf{A}$ has a factorization $\mathbf{A} = \mathbf{H}\mathbf{U}$, where $\mathbf{H}$ is Hermitian und $\mathbf{U}$ is unitary, hence both factors are equivalent to diagonal matrices. On the other hand, the value of φ is the same for equivalent matrices, just as the value of the determinant, since

$$\varphi\mathbf{A} = \varphi(\mathbf{B}\mathbf{B}^{-1}\mathbf{A}) = \varphi\mathbf{B}(\varphi\mathbf{B}^{-1})\varphi\mathbf{A} = \varphi\mathbf{B}\varphi\mathbf{A}\varphi\mathbf{B}^{-1} = \varphi(\mathbf{B}\mathbf{A}\mathbf{B}^{-1}).$$

So $\mathbf{A}$ is a product of two function values depending on diagonal matrices, hence it depends only on a diagonal matrix $\mathbf{D}$ having the same determinant as $\mathbf{A}$ since also the determinant is a multiplicative function. Therefore, considering the factorization

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & & \\ 0 & d_2 & 0 & \cdots & \\ 0 & 0 & d_3 & 0 & \cdots \\ & & \cdots & & \end{bmatrix} = \begin{bmatrix} d_1 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & & \\ 0 & d_2 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots \\ & \cdots & & & \end{bmatrix} \cdots =$$

$$= \prod_{k=1}^{n} \mathbf{P}_k \begin{bmatrix} d_k & 0 & \cdots & \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{bmatrix} \mathbf{P}_k^{-1},$$

where $\mathbf{P}_k$ consists of the elements of the unit matrix, but the first and kth rows are permuted, we get

$$\varphi\mathbf{A}=\varphi\mathbf{D}=\prod_{k=1}^{n}\varphi\begin{bmatrix} d_k & 0 & \cdots & \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{bmatrix}=\varphi\begin{bmatrix} \prod_{k=1}^{n} d_k & 0 & \cdots & \\ 0 & 1 & 0 & \cdots \\ & \cdots & & \end{bmatrix}=$$

$$=f(\det\mathbf{D})=f(\det\mathbf{A})$$

for every $\mathbf{A}\in K_n^{\times}$.

The theorem proved above gives a possibility of axiomatizing determinants without coordinates[1]) by the multiplicativity and by the homogeneity:

$$\varphi(\lambda\mathbf{A})=\lambda^n\varphi\mathbf{A},\quad \lambda\in K,\ \mathbf{A}\in K_n^{\times},$$

e.g., if K is the real field and n is odd.[2])

As a corollary we get as characteristic properties of the determinant the multiplicativity and the additivity for a column and row vector, respectively. These properties were used by M. Stojaković [7] to characterize the determinant.

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[1]) The problem of characterizing determinants without coordinates has arised in [1—2].

[2]) J. Gáspár [3] could characterize Dieudonné's determinant over a sfield by the multiplicativity and by the homogeneity.