

On the distribution of zeros of general exponential polynomials.

To O. Varga, to his 50th birthday.

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1. Starting from the theory of the zeta-function of Riemann I was led in 1949 to the question to give an upper bound for the number $N(f, a, a+d)$ of the zeros (counted according to their multiplicity) in the real interval

$$(1.1) \quad a \leq x \leq a+d \quad a > 0, d > 0$$

of the almost-periodical polynomial

$$f(x) = \sum_{r=0}^n a_r \cos \lambda_r x$$

with

$$0 = \lambda_0 < \lambda_1 < \dots < \lambda_n, \quad a_n > 0,$$

depending only upon λ_n, n, d and a . I have found¹⁾ by methods treated later in my book²⁾ the inequality

$$(1.2) \quad N(f, a, a+d) < 6d\lambda_n + 6n \log \left(24 \frac{a+d}{d} \right).$$

In case of

$$(1.3) \quad f^*(x) = \sum_{r=1}^n \frac{\cos(x \log r)}{\sqrt{r}}$$

$$(1.4) \quad a^* = 2\pi n^2, \quad a^* + d^* = 2\pi(n+1)^2$$

which was the starting point of these investigations, (1.2) gives the estimation

$$(1.5) \quad N(f^*, a^*, d^*) < (24\pi + 6)n \log n + O(n),$$

while an estimation

$$N(f^*, a^*, d^*) < c n \log n + O(n)$$

¹⁾ On the distribution of real roots of almost-periodical polynomials, *Publ. Math. Debrecen* **1** (1949), 38—41.

²⁾ Eine neue Methode in der Analysis und deren Anwendungen, *Budapest*, 1953.

with $c < 4$ would give by a simple reasoning that the main term of the so-called Riemann-Siegel asymptotical formula for $\zeta\left(\frac{1}{2} + ix\right)$ has in the interval $0 < x \leq T$ for $T \rightarrow +\infty$ at least $c_1 T \log T$ real zeros (c_1 positive numerical constant). Now Mr. T. GANELIUS³⁾ kindly informed me about the following theorem of PÓLYA.⁴⁾

If $z = x + iy$

$$\mu_1 < \mu_2 < \dots < \mu_l$$

and $P_\nu(z)$ is for $\nu = 1, 2, \dots, l$ a polynomial of degree $\leq m_\nu - 1$ with

$$m_1 + m_2 + \dots + m_l = n$$

$$P_1(z)P_l(z) \neq 0,$$

then the number $N_1(g_l, a, a+d)$ of the zeros (according to multiplicity) of the function

$$(1.6) \quad g_l(z) = \sum_{\nu=1}^l P_\nu(z) e^{i\mu_\nu z}$$

in the infinite vertical strip

$$a \leq x \leq a+d$$

satisfies the inequality

$$(1.7) \quad \left| N_1(g_l, a, a+d) - \frac{(\mu_l - \mu_1)d}{2\pi} \right| \leq n-1.$$

This estimation supersedes (1.2) in all respects; in the case (1.3)—(1.4) it gives the estimation

$$N_1(f^*, a^*, a^* + d^*) = 4n \log n + O(n),$$

as easy to see. However a theorem like (1.7) cannot be expected to hold for the more general class

$$(1.8) \quad G_l(z) = \sum_{\nu=1}^l P_\nu(z) e^{\omega_\nu z}$$

with arbitrary complex exponents ω_ν or even for the subclass

$$(1.9) \quad G_l(z) = \sum_{\nu=1}^l b_\nu e^{\omega_\nu z}$$

with arbitrary complex coefficients b_ν with

$$b_1 b_2 \dots b_{l-1} b_l \neq 0.$$

³⁾ In a letter from 25. Jan. 1957.

⁴⁾ *Jber. Deutsch. Math. Verein.* 3 (1925) p. 97, Problem 24. Solution in Bd. 37 (1928) p. 83 by N. OBRESCHKOFF.

Nevertheless PÓLYA⁵⁾ proved for the number $N_2(G_l, r)$ of zeros (according to multiplicity) of $G_l(z)$ for $|z| \leq r$ the formula

$$N_2(G_l, r) = Er + O(\log r)$$

with a positive E , independent upon r and even⁶⁾

$$(1.10) \quad N_2(G_l, r) = Er + O(1).$$

Now we are going to prove with the methods of my book the following theorem on the distribution of the zeros of $G_n(z)$, which is obviously not covered by any of the above named results.

Theorem. *If*

$$(1.11) \quad \max_j |\omega_j| \leq M$$

and

$$(1.12) \quad \min_{\mu \neq \nu} |\omega_\mu - \omega_\nu| \geq A (> 0),$$

then the number of zeros of $G_n(z) = \sum_{\nu=1}^n b_\nu e^{\omega_\nu z} \not\equiv 0$ in the square

$$(1.13) \quad \begin{aligned} A &\leq x \leq A + D \\ -\frac{D}{2} &\leq y \leq \frac{D}{2} \\ (A > 0, D > 0) \end{aligned}$$

(according to multiplicity) is not greater than

$$(1.14) \quad 5MD + \log 2n + n \log \left(2 + \frac{2n^3}{AD} \right).$$

This upper bound is independent upon the position of the square, upon the coefficients b_ν and as to the exponents it depends only upon M and A . As the trivial example $G_2(z) = e^{iMz} - 1$ shows, the constant 5 in (1.14) cannot be replaced by any other $< \frac{1}{2\pi}$, further the example

$$\left(e^{\frac{i}{n-1}x} - 1 \right)^{n-1}$$

with a large integer n shows that the dependence upon n , the number of

⁵⁾ Münchener Sitzungsber. (1920) p. 285—290.

⁶⁾ Announced in his paper: Lücken und Singularitäten von Potenzreihen, *Math. Z.* **29** (1929), 549—560; in particular footnote on p. 594.

terms, is also indispensable and the dependence of our upper bound (1.14) upon n is also not very bad. The only question which remains open is whether or not the dependence upon d is superfluous.

2. As to the proof it depends ultimately upon a theorem which played in the German version of my book⁷⁾ a subordinate role but in the Chinese edition⁸⁾ and in the forthcoming completely rewritten English edition its role is essentially increased. We shall state only its most important corollary⁹⁾ in an apparently more general form.

If m is a prescribed nonnegative integer and with a positive δ we have

$$(2.1) \quad \frac{\min_{\mu \neq \nu} |z_\mu - z_\nu|}{\max_j |z_j|} \geq \delta,$$

then with these z_j 's to an arbitrary system $(c_1, c_2, \dots, c_n)$ of complex numbers there is an integer r with

$$(2.2) \quad m+1 \leq r \leq m+n$$

and

$$(2.3) \quad \frac{|c_1 z_1^r + \dots + c_n z_n^r|}{|c_1| |z_1|^r + \dots + |c_n| |z_n|^r} \geq \frac{1}{2n} \left(\frac{\delta}{2} \right)^{n-1},$$

independently upon m .

We shall transform it to our present aims. Let with $a > 0, d > 0$ be

$$m = \left[\frac{an}{d} \right], \quad z_j = e^{\frac{d}{n} w_j} \quad (j = 1, 2, \dots, n)$$

then the above theorem asserts that from

$$(2.4) \quad \frac{\min_{j \neq l} \left| e^{\frac{d}{n} w_j} - e^{\frac{d}{n} w_l} \right|}{\max_j \left| e^{\frac{d}{n} w_j} \right|} \geq \delta,$$

it follows the existence of an integer r with

$$(2.5) \quad a \leq \frac{dr}{n} \leq a+d$$

⁷⁾ L. c. ²⁾ p. 53.

⁸⁾ Peking, 1956.

⁹⁾ See l. c. ²⁾ p. 56.

such that

$$\frac{\left| \sum_{j=1}^n c_j e^{w_j \frac{d}{n}} \right|}{\sum_{j=1}^n |c_j| \left| e^{w_j \frac{d}{n}} \right|} \geq \frac{1}{2n} \left(\frac{\delta}{2} \right)^{n-1},$$

i. e. from (2.5) a fortiori the inequality

$$(2.6) \quad \max_{a \leq t \leq a+d} \frac{\left| \sum_{j=1}^n c_j e^{w_j t} \right|}{\sum_{j=1}^n |c_j| \left| e^{w_j t} \right|} \geq \frac{1}{2n} \left(\frac{\delta}{2} \right)^{n-1}$$

follows. This is what we need in the sequel.

3. Now we turn to the proof of our theorem. Let for

$$1 \leq j \neq l \leq n$$

be

$$\omega_j - \omega_l = |\omega_j - \omega_l| e^{i\varphi_{jl}}, \quad -\pi < \varphi_{jl} \leq \pi;$$

this gives $n(n-1)$ vectors so that with each vector also its negative occurs. Hence there is an α with $-\pi < \alpha \leq \pi$ so, that the open angles with the respective bisectrices $\alpha \pm \frac{\pi}{2}$ and both with the opening $\frac{2\pi}{n(n-1)}$ do not contain any of our vectors. Hence for each (j, l) -pair we have either

$$(3.1) \quad -\frac{\pi}{2} + \frac{\pi}{n(n-1)} \leq \varphi_{jl} - \alpha \leq \frac{\pi}{2} - \frac{\pi}{n(n-1)} \quad \text{mod } (-\pi, \pi]$$

or

$$(3.2) \quad \frac{\pi}{2} + \frac{\pi}{n(n-1)} \leq \varphi_{jl} - \alpha \leq \frac{3\pi}{2} - \frac{\pi}{n(n-1)} \quad \text{mod } (-\pi, \pi].$$

In the case (3.1) we have with this α

$$(3.3) \quad \begin{aligned} & \left| e^{\frac{D}{2n} e^{-i\alpha}(\omega_j - \omega_l)} - 1 \right| = \left| e^{\frac{D}{2n} |\omega_j - \omega_l| e^{i(\varphi_{jl} - \alpha)}} - 1 \right| \geq \\ & \geq e^{\frac{D}{2n} |\omega_j - \omega_l| \cos(\varphi_{jl} - \alpha)} - 1 \geq e^{\frac{D}{2n} D \sin \frac{\pi}{n(n-1)}} - 1 > \\ & > \frac{DA}{2n} \sin \frac{\pi}{n(n-1)} > \frac{DA}{n^3} > \frac{DA}{n^3 + DA}. \end{aligned}$$

In the case (3.2) we have

$$\begin{aligned}
 (3.4) \quad & \left| e^{\frac{D}{2n} e^{-i\alpha} (\omega_j - \omega_l)} - 1 \right| = \left| 1 - e^{\frac{D}{2n} |\omega_j - \omega_l| e^{i(\varphi_{jl} - \alpha)}} \right| \geq \\
 & \geq 1 - e^{-\frac{D}{2n} A \sin \frac{\pi}{n(n-1)}} \geq \frac{\frac{D}{2n} A \sin \frac{\pi}{n(n-1)}}{1 + \frac{D}{2n} A \sin \frac{\pi}{n(n-1)}} \geq \\
 & \geq \frac{\frac{DA}{n^3}}{1 + \frac{DA}{n^3}} = \frac{DA}{n^3 + DA}
 \end{aligned}$$

too. Hence

$$(3.5) \quad \frac{\min_{j \neq l} \left| e^{\frac{D}{2n} e^{-i\alpha} \omega_j} - e^{\frac{D}{2n} e^{-i\alpha} \omega_l} \right|}{\max_j \left| e^{\frac{D}{2n} e^{-i\alpha} \omega_j} \right|} \geq \frac{DA}{n^3 + DA} \cdot \frac{\min_j \left| e^{\frac{D}{2n} \omega_j e^{-i\alpha}} \right|}{\max_j \left| e^{\frac{D}{2n} \omega_j e^{-i\alpha}} \right|} \geq \frac{DA}{n^3 + DA} e^{-\frac{D}{n} M}.$$

Now we apply (2.4)–(2.6) with

$$\begin{aligned}
 c_j &= b_j e^{\omega_j \left(A + \frac{D}{2}\right)} \\
 w_j &= \omega_j e^{-i\alpha} \\
 a &= 0, \quad d = \frac{D}{2}.
 \end{aligned}$$

Then (3.5) means that (2.4) is satisfied with

$$(3.6) \quad \delta = \frac{DA}{n^3 + DA} e^{-\frac{DM}{n}}.$$

Hence (2.6) gives

$$(3.7) \quad \max_{0 \leq t \leq \frac{D}{2}} \frac{\left| \sum_{j=1}^n b_j e^{\omega_j \left(A + \frac{D}{2}\right)} e^{\omega_j e^{-i\alpha} t} \right|}{\sum_{j=1}^n \left| b_j e^{\omega_j \left(A + \frac{D}{2}\right)} \right| \left| e^{\omega_j e^{-i\alpha} t} \right|} \geq \frac{1}{2n} \left(\frac{DA}{2(n^3 + DA)} e^{-\frac{DM}{n}} \right)^{n-1}.$$

If the max on the left is attained for $t = t_0$ and

$$A + \frac{D}{2} + t_0 e^{-i\alpha} = z_0; \quad 0 \leq t_0 \leq \frac{D}{2},$$

then (3.7) gives

$$(3.8) \quad \frac{|G_n(z_0)|}{\sum_{j=1}^n |b_j| |e^{\omega_j z_0}|} \geq \frac{1}{2n} \left(\frac{D\mathcal{A}}{2(n^3 + D\mathcal{A})} e^{-\frac{D}{n}M} \right)^{n-1}.$$

4. Having once the inequality (3.8) the proof can be easily completed by applying Jensen's estimation to $G_n(z)$ and the circle $|z - z_0| \leq D\sqrt{2}$ (which obviously covers our square (1.13)). Hence the number of zeros of $G_n(z)$ in the square (1.13) is

$$\leq \max_{|z - z_0| \leq D\sqrt{2}} \log \left| \frac{G_n(z)}{G_n(z_0)} \right| = \max_{|w| \leq D\sqrt{2}} \log \left| \frac{G_n(z_0 + w)}{G_n(z_0)} \right|$$

and using (3.8) and (1.11)

$$\begin{aligned} &< \max_{|w| \leq D\sqrt{2}} \log \left\{ \left(\sum_{j=1}^n |b_j e^{\omega_j z_0}| |e^{\omega_j w}| \right) 2n \left(\frac{2(n^3 + D\mathcal{A})}{D\mathcal{A}} e^{\frac{D}{n}M} \right)^n \frac{1}{\sum_{j=1}^n |b_j| |e^{\omega_j z_0}|} \right\} \leq \\ &\leq (e\sqrt{2} + 1)MD + \log 2n + n \log \left(2 + \frac{2n^3}{D\mathcal{A}} \right). \end{aligned}$$

Q. e. d.

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