

The Cauchy and Jensen differences on semigroups

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Abstract. Let $(S, +)$ be a commutative and uniquely divisible by 2 semigroup with the neutral element, $(G, +)$ be a topological group (not necessarily commutative), and K be a normal discrete subgroup of G . We prove that if S is endowed with a suitable topology, then every function $f : S \rightarrow G$, continuous at a point and satisfying the condition:

$$(1) \quad f(x+y) - f(x) - f(y) \in K \quad \text{for every } x, y \in S,$$

admits the representation $f = k + A$ with some function $k : S \rightarrow K$ and an additive function $A : S \rightarrow G$.

We also study functions $h : S \rightarrow G$ fulfilling

$$(2) \quad 2h\left(\frac{x+y}{2}\right) - h(x) - h(y) \in K \quad \text{for every } x, y \in S$$

as well as the situations where (1) and (2) hold almost everywhere in S^2 with respect to some ideals.

1. Introduction and preliminary facts

Throughout this paper we assume the following hypotheses:

- (A₁) $(S, +)$ is a commutative semigroup uniquely divisible by 2 (i.e. for every $x \in S$ there is exactly one $y \in S$ with $x = 2y$) with the neutral element 0;
- (A₂) $(G, +)$ is a topological group (not necessarily commutative);
- (A₃) K is a normal discrete subgroup of G (discrete means that there is a neighbourhood $U \subset G$ of zero such that $U \cap K = \{0\}$).

We are going to investigate functions $f : S \rightarrow G$ satisfying the condition

$$(1.1) \quad f(x+y) - f(x) - f(y) \in K \quad \text{for every } x, y \in S.$$

It is proved (see e.g. [1]–[4]) that if S is a real topological linear space, then, under some additional assumptions, there exists an additive function $A : S \rightarrow G$ (i.e. $A(x + y) = A(x) + A(y)$ for every $x, y \in S$) such that

$$(1.2) \quad f(x) - A(x) \in K \quad \text{for every } x \in S,$$

which means that f has the representation $f = k + A$ with some function $k : S \rightarrow K$. In general, i.e. without any additional assumptions, the statement is not true; this results e.g. from an example of G. GODINI (cf. [1]–[3]).

We show that if there is given a suitable topology in S , then every function $f : S \rightarrow G$ continuous at a point and satisfying (1.1) admits the representation which is described above (cf. [5]). We also study functions $h : S \rightarrow G$ fulfilling

$$(1.3) \quad 2h\left(\frac{x+y}{2}\right) - h(x) - h(y) \in K \quad \text{for every } x, y \in S$$

as well as the situations where (1.2) and (1.3) hold almost everywhere with respect to some ideals.

Let us start with some definitions and examples.

Throughout the paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ stand for the sets of all positive integers, integers, rationals, and reals, respectively. We also need the following four hypotheses:

(H₀) S is endowed with a topology.

(H₁) For every $y \in S$ the translation $T_y : S \rightarrow S$, given by: $T_y(x) = x + y$ is continuous at 0.

(H₂) For every neighbourhood $U \subset S$ of 0 there is a set $V \subset U$ with

(i) $\frac{x}{2} \in V$ for every $x \in V$;

(ii) $S = \bigcup \{2^n V : n \in \mathbb{N}\}$.

(H₃) For every neighbourhood $U \subset S$ of 0 there is a neighbourhood $V \subset U$ of 0 such that conditions (i), (ii) of (H₂) are valid.

Note that (H₃) implies (H₂); moreover, (ii) and (A₁) yield $0 \in V$.

Now, we will give some examples of spaces satisfying the above hypotheses. For instance, every topological locally J-convex connected commutative group (see [5], Remark 1) or every topological linear space fulfils (H₃). The additive semigroups $[0, +\infty)$ and $\{2^{-m}n : n, m \in \mathbb{N} \cup \{0\}\}$, with the topologies induced from $\mathbb{R}$, also satisfy (H₃). We will show that (H₂) holds for every semitopological linear space. Moreover, Example 1.1

proves that there are semitopological linear spaces which do not satisfy (H₃). (Let us remind that a real linear space is called semitopological provided it is endowed with a semilinear topology, i.e. a topology such that the mapping $L : \mathbb{R} \times X \times X \rightarrow X$, $L(a, x, y) = ax + y$ is separately continuous with respect to either variable. For further details concerning semilinear topologies we refer e.g. to [10].)

Let X be a semitopological real linear space and $U \subset X$ be a neighbourhood of the origin. For every $x \in X$ there is $a_x \in \mathbb{R}$, $a_x > 0$, such that $bx \in U$ for every $b \in (-a_x, a_x)$. Put $V = \bigcup \{(-a_x, a_x)x : x \in X\}$, where $(-a_x, a_x)x = \{ax : a \in (-a_x, a_x)\}$. It is easily seen that (i) and (ii) are valid.

Example 1.1. Let $S = \mathbb{R}^2$ be endowed with the core topology (cf. [10]); i.e. a set $A \subset S$ is open iff every $x \in A$ is algebraically interior to A (a point $x \in A$ is algebraically interior to A provided for every $y \in S \setminus \{(0, 0)\}$ there is $a_y \in \mathbb{R}$, $a_y > 0$, such that $x + by \in A$ for all $b \in (-a_y, a_y)$). Then S is a semitopological real linear space (cf. [10], p. 596).

Fix $a \in \mathbb{R}$, $a > 1$, and put

$$U = \left\{ (x, y) \in \mathbb{R}^2 : y \notin \left[\frac{1}{a}x^2, ax^2 \right] \right\} \cup \{(0, 0)\}.$$

It is easily seen that $U \setminus \{(0, 0)\}$ is open in the usual topology in $\mathbb{R}^2$ and therefore it is open in the core topology (see [10], Corollary 1). Further, $(0, 0)$ is algebraically interior to U . Thus U is an open neighbourhood of $(0, 0)$ in the core topology. Let $V \subset U$ be a neighbourhood of $(0, 0)$ (in the core topology), which means that $(0, 0)$ is algebraically interior to $\text{int}_c V$ ($\text{int}_c V$ denotes the interior of V with respect to the core topology). Hence there is $x \in \mathbb{R}$, $a > x > 0$, such that $(x, 0) \in \text{int}_c V$, whence there is $y \in \mathbb{R}$, $0 < y < \frac{1}{a}x^2$, with $(x, y) \in \text{int}_c V$. Note that if $a^2 > 2$, then there exists $n \in \mathbb{N}$ with $\frac{1}{a}x^2 < 2^n y < ax^2$ and consequently

$$\frac{1}{a}(2^{-n}x)^2 < 2^{-n}y < a(2^{-n}x)^2.$$

So $(2^{-n}x, 2^{-n}y) \notin U$, which yields $(2^{-n}x, 2^{-n}y) \notin V$. It means that condition (i) is not valid (when $a^2 > 2$). In this way we have proved that $S = \mathbb{R}^2$ endowed with the core topology does not satisfy (H₃).

In the whole paper, in the factor group G/K we assume the factor topology: a set $U \subset G/K$ is open iff the set $p^{-1}(U)$ is open in G , where $p : G \rightarrow G/K$ is the natural projection. G/K endowed with this topology is a topological group.

Now, let us recall some results from [4] and [5], which will be useful in the sequel.

Lemma 1.2 (see [5], Lemma 2). *Suppose that (A_1) holds and $V \subset S$ is a set such that conditions (i), (ii) of (H_2) are valid. Let $(Y, +)$ be a group (not necessarily commutative) and $f : V \rightarrow Y$ be a function satisfying*

$$f(x + y) = f(x) + f(y) \quad \text{for every } x, y \in V \text{ with } x + y \in V.$$

Then there exists a unique additive function $g : S \rightarrow Y$ which is an extension of f .

(Actually Lemma 2 in [5] is formulated for V being a neighbourhood of 0, but its proof is also suitable for our Lemma 1.2.)

Lemma 1.3 (see [4], Lemma 1). *Suppose (A_2) and (A_3) . Let X be a topological space and let $g : X \rightarrow G/K$ be a function continuous at a point $x_0 \in X$. Then there exists a function $k : X \rightarrow G$ continuous at x_0 such that $k(x) \in g(x)$ for every $x \in X$.*

The next theorem generalizes some known results concerning the Jensen functional equation, i.e. the equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

It is easily seen that if G is uniquely divisible by 2, then the Jensen equation and the equation

$$(1.4) \quad 2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$$

have the same sets of solutions in the class of functions $f : D \rightarrow G$ for every $D \subset S$ which is J-convex (i.e. $\frac{x+y}{2} \in D$ for every $x, y \in D$).

Theorem 1.4. *Let S be as in hypothesis (A_1) , $D \subset S$ be a J-convex set, and $(Y, +)$ be a group (not necessarily commutative). Suppose that a function $f : D \rightarrow Y$ satisfies equation (1.4) and there is $y_0 \in D$ such that condition (ii) of (H_2) holds with $V = \{y \in S : y + y_0 \in D\}$. Then there are, uniquely determined, an additive mapping $g : S \rightarrow Y$ and an element $u \in Y$ such that*

$$(1.5) \quad f(x) = g(x) + u \quad \text{for every } x \in D.$$

PROOF. First, let us observe that V is J-convex, $0 \in V$, and according to (1.4), for every $x, y \in D$

$$(1.6) \quad f(x) + f(y) = 2f\left(\frac{x+y}{2}\right) = 2f\left(\frac{y+x}{2}\right) = f(y) + f(x)$$

and consequently

$$(1.7) \quad \begin{aligned} f(x) - f(y) &= -f(y) + f(y) + f(x) - f(y) \\ &= -f(y) + f(x) + f(y) - f(y) = -f(y) + f(x). \end{aligned}$$

Define a function $h : V \rightarrow Y$ by:

$$h(x) = f(x + y_0) - f(y_0) \quad \text{for every } x \in V.$$

Then $h(0) = 0$ and, by (1.4), (1.6) and (1.7), for every $x, y \in V$,

$$\begin{aligned} 2h\left(\frac{x+y}{2}\right) &= 2\left(f\left(\frac{x+y_0+y+y_0}{2}\right) - f(y_0)\right) \\ &= 2f\left(\frac{x+y_0+y+y_0}{2}\right) - 2f(y_0) = f(x+y_0) + f(y+y_0) - 2f(y_0) \\ &= h(x) + h(y). \end{aligned}$$

Thus

$$(1.8) \quad 2h\left(\frac{x}{2}\right) = 2h\left(\frac{x+0}{2}\right) = h(x) \quad \text{for } x \in V$$

and, for every $x, y \in V$ with $x+y \in V$,

$$h(x+y) = 2h\left(\frac{x+y}{2}\right) = h(x) + h(y).$$

Further, since $0 \in V$ and V is J-convex, condition (i) of (H_2) holds. Hence, on account of Lemma 1.2, there exists a unique additive function $g : S \rightarrow Y$ with $h(x) = g(x)$ for $x \in V$. Consequently, according to the definition of h ,

$$(1.9) \quad f(x) = g(x) - g(y_0) + f(y_0) = g(x) + u \quad \text{for } x \in V + y_0,$$

where $u = -g(y_0) + f(y_0)$. There are $m \in \mathbb{N}$ and $z \in V$ with $y_0 = 2^m z$. Thus

$$\begin{aligned} u &= -2^m g(z) + f(y_0) = -2^m h(z) + f(y_0) \\ &= -2^m (f(z + y_0) - f(y_0)) + f(y_0), \end{aligned}$$

which, by (1.6) and (1.7), yields

$$(1.10) \quad f(x) - u = -u + f(x) \quad \text{for every } x \in D.$$

We will show by induction that

$$(1.11) \quad 2^n (f(2^{-n}y + (1 - 2^{-n})x) - u) = f(y) + (2^n - 1)f(x) - 2^n u$$

for every $x, y \in D$ and $n \in \mathbb{N}$. For this, fix $x, y \in D$. The case $n = 1$ is trivial. So suppose that the equality is valid for some $n \in \mathbb{N}$. Then, on account of (1.4), (1.6) and (1.10),

$$\begin{aligned}
& 2^{n+1}[f(2^{-n-1}y + (1 - 2^{-n-1})x) - u] \\
&= 2^{n+1} \left[f \left(\frac{1}{2}(2^{-n}y + (1 - 2^{-n})x + x) \right) - u \right] \\
&= 2^n f(2^{-n}y + (1 - 2^{-n})x) + 2^n f(x) - 2^{n+1}u \\
&= 2^n [f(2^{-n}y + (1 - 2^{-n})x) - u] + 2^n f(x) - 2^n u \\
&= f(y) + (2^n - 1)f(x) - 2^n u + 2^n f(x) - 2^n u \\
&= f(y) + (2^{n+1} - 1)f(x) - 2^{n+1}u.
\end{aligned}$$

Thus, by induction, we have shown that (1.11) holds for every $n \in \mathbb{N}$ and $x, y \in D$.

Take $y \in D$. According to the fact that V fulfils conditions (i), (ii) of (H_2) , there is $m \in \mathbb{N}$ with

$$(1.12) \quad 2^{-m}y_0, 2^{-m-1}y_0, 2^{-m}y + 2^{-m-1}y_0 \in V.$$

Further, $0 \in V$, whence by induction, the J-convexity of V yields

$$(1 - 2^{-n})w \in V \quad \text{for every } n \in \mathbb{N}, w \in V.$$

Consequently from (1.12) we get

$$\begin{aligned}
& 2^{-m-1}y + (2^{-m-1} - 2^{-2m-1})y_0 \\
&= \frac{1}{2}((2^{-m}y + 2^{-m-1}y_0) + (1 - 2^{-m+1})2^{-m-1}y_0) \in V,
\end{aligned}$$

which means that

$$2^{-m-1}y + (1 - 2^{-m-1})(1 + 2^{-m})y_0 \in V + y_0.$$

Hence, by virtue of (1.9) and additivity of g ,

$$\begin{aligned}
& 2^{m+1}[f(2^{-m-1}y + (1 - 2^{-m-1})(1 + 2^{-m})y_0) - u] \\
&= g(y) + (2^{m+1} - 1)g((1 + 2^{-m})y_0).
\end{aligned}$$

This equality, jointly with (1.11) (for $x = (1 + 2^{-m})y_0 \in V + y_0$ and $n = m + 1$) and (1.9) (for $x = (1 + 2^{-m})y_0 \in V + y_0$), gives

$$\begin{aligned} f(y) &= 2^{m+1}[f(2^{-m-1}y + (1 - 2^{-m-1})(1 + 2^{-m})y_0) - u] \\ &\quad - (2^{m+1} - 1)f((1 + 2^{-m})y_0) + 2^{m+1}u \\ &= g(y) + (2^{m+1} - 1)g((1 + 2^{-m})y_0) - (2^{m+1} - 1)f((1 + 2^{-m})y_0) \\ &\quad + 2^{m+1}u \\ &= g(y) - (2^{m+1} - 1)u + 2^{m+1}u = g(y) + u. \end{aligned}$$

Finally suppose that

$$(1.13) \quad g_1(x) + u_1 = f(x) = g_2(x) + u_2 \quad \text{for } x \in D$$

for additive functions $g_1, g_2 : S \rightarrow Y$ and $u_1, u_2 \in Y$. Then

$$g_1(y) + g_1(y_0) + u_1 = g_2(y) + g_2(y_0) + u_2 \quad \text{for } y \in V,$$

from which (with $y = 0$) we derive $g_1(y_0) + u_1 = g_2(y_0) + u_2$ and consequently $g_1(y) = g_2(y)$ for $y \in V$, where V and y_0 are just the same as at the beginning of the proof. Hence, according to Lemma 1.2, $g_1 = g_2$ and by (1.13), $u_1 = u_2$. This completes the proof.

2. The Cauchy and Jensen differences of functions continuous at a point

In this part we generalize slightly some results from [1], [4], and [5]. Let us begin with the following

Theorem 2.1 (cf. [5], Theorem 1). *Suppose (A_1) – (A_3) , (H_0) , and (H_2) . Let $V \subset S$ be a neighbourhood of 0 and $f : V \rightarrow G$ be a function continuous at 0, satisfying*

$$(2.1) \quad f(x + y) - f(x) - f(y) \in K \quad \text{for every } x, y \in V \text{ with } x + y \in V.$$

Then there exists an additive function $A : S \rightarrow G$ such that

$$f(x) - A(x) \in K \quad \text{for every } x \in V.$$

If, moreover, (H_3) is valid, then, in a unique way, A can be chosen continuous at 0.

THE PROOF of this theorem is almost the same as the proof of Theorem 1 in [5] (cf. also [1], the proof of Theorem 3); therefore we omit it.

There is only one difference. Namely, supposing more general hypothesis (H_2) instead of (H_3) (in [5] it is denoted by (H_2)) we cannot get continuity of A at 0. The reason is that by a suitable definition of A , in the proof of Theorem 1 from [5] we have $f(x) = A(x)$ for every x belonging to some neighbourhood $V \subset S$ of 0 satisfying conditions (i) and (ii). Under our hypothesis (H_2) we get the same relation, but only for a set $V \subset S$ such that (i) and (ii) are valid, which does not need to imply continuity of A at 0 (cf. Example 2.7).

Theorem 2.2. *Suppose (A_1) – (A_3) and (H_0) – (H_2) . Let $F : S \rightarrow G/K$ be an additive function continuous at a point $x_0 \in S$. Then there exists an additive function $A : S \rightarrow G$ such that $A(x) \in F(x)$ for every $x \in S$. If, moreover, (H_3) holds, then, in a unique way, A can be chosen continuous at 0.*

PROOF. Fix a neighbourhood $U \subset G/K$ of zero. Since $U + F(x_0)$ is a neighbourhood of $F(x_0)$, there is a neighbourhood $V \subset S$ of x_0 with $F(V) \subset U + F(x_0)$. Let $W \subset S$ be a neighbourhood of 0 such that $W + x_0 \subset V$. Then we have

$$F(W) + F(x_0) = F(W + x_0) \subset F(V) \subset U + F(x_0),$$

which means that $F(W) \subset U$.

In this way we have proved that F is continuous at 0. According to Lemma 1.3, there is a function $k : S \rightarrow G$ continuous at 0, such that $k(x) \in F(x)$ for every $x \in S$. Note that $k(x + y) - k(x) - k(y) \in K$ for every $x, y \in S$. Hence Theorem 2.1 yields the statement.

From Theorem 2.2 we get the following generalization of Theorem 2.1.

Corollary 2.3. *Let G , K , and S be the same as in Theorem 2.2 and $f : S \rightarrow G$ be a function continuous at a point $x_0 \in S$, satisfying (1.1). Then the statement of Theorem 2.1 holds with $V = S$.*

PROOF. It is easy to see that the function $F = p \circ f$ (where $p : G \rightarrow G/K$ is the natural projection) is additive and continuous at x_0 . Thus Theorem 2.2 implies the assertion.

Remark 2.4. The assumption of Theorem 2.2 and Corollary 2.3 that the topology in S fulfils (H_2) (in particular (ii)) is essential as it results from Remark 3 in [5].

In the general situation we cannot get continuity of A at $x_0 \neq 0$. This is shown by the two examples given below.

Example 2.5. Let $a, b \in \mathbb{R}$, $a > 0$, $b > 0$, $ab^{-1} \notin \mathbb{Q}$. Put $K = \{0\}$ and

$$S = \{pa + qb : p, q \in \mathbb{Q}, p \geq 0, q \geq 0\}$$

and define a function $A : S \rightarrow \mathbb{R}$ by

$$A(pa + qb) = p \quad \text{for every } q, p \in \mathbb{Q}, p \geq 0, q \geq 0.$$

Suppose that S is endowed with the topology induced by $\mathbb{R}$. Then (H_1) , (H_3) hold and A is additive, continuous at 0, and discontinuous at every $x \in S \setminus \{0\}$.

Example 2.6. Let $(G, +) = (\mathbb{R}, +)$ with the usual topology, $(S, +) = ([0, +\infty), +)$ with the topology generated by the following family of open sets:

$$\begin{aligned} T_0 = & \{[a, b) + k\mathbb{N} : k \in \mathbb{N}, a, b \in (0, +\infty), a < b\} \\ & \cup \{[a, b) : a, b \in [0, 1), a < b\}, \end{aligned}$$

and $K = \mathbb{Z}$. Then it is easy to check that the topology in S is Hausdorff and (H_1) and (H_3) are valid. Put

$$f(x) = x - [x] \quad \text{for } x \in S,$$

where $[x] = \max\{k \in \mathbb{Z} : k \leq x\}$. Then $f : S \rightarrow G$ is continuous and satisfies (1.1). Further, the function $A : S \rightarrow G$ given by $A(x) = x$ for $x \in S$ is additive, continuous at 0, and discontinuous at every point $x \in S$, $x \geq 1$, because every neighbourhood of a point $x \in S$ with $x \geq 1$ contains a subset of the form $[x, b) + k\mathbb{N}$ for some $b \in (x, +\infty)$ and $k \in \mathbb{N}$. Since every additive function $A_0 : S \rightarrow G$ continuous at a point must be continuous at 0 (in view of (H_1)), by virtue of Corollary 2.3 A is the only function additive and continuous at a point which satisfies (1.2).

The next example shows that assuming only (H_1) and (H_2) we cannot even get continuity of A at 0.

Example 2.7. Let G , K , S and f be the same as in Example 2.6 with one exception; this time we generate a topology in S by the following family of open sets:

$$T = \{[a, b) + k\mathbb{N} : k \in \mathbb{N}, a, b \in [0, +\infty), a < b\}.$$

Then (H_1) and (H_2) are valid and f is continuous and fulfils (1.1). Since every additive function $A : S \rightarrow G$ continuous at 0 (in the topology generated by T) is continuous at 0 in the usual topology in $[0, +\infty)$, the

function $A_0 : S \rightarrow G$, $A_0 \equiv 0$ is the only additive function continuous at 0 (with respect to the topology generated by T) mapping S into G . Of course it is not true that $f(x) - A_0(x) \in K$ for every $x \in S$.

The last example in this part shows that we cannot get uniqueness of A (in Theorem 2.1), in the class of all additive functions mapping S into G , if G is “only” a topological group. However, the situation is completely different under some additional assumptions and this will be proved in the proposition below, but first let us see the following

Example 2.8. In $G = \mathbb{R}^2$ we introduce a topology in the following way: a set $D \subset G$ is open iff, for every $y \in \mathbb{R}$, the set $\{x : (x, y) \in D\}$ is open in $\mathbb{R}$ (with the usual topology). Then G is an additive topological group and $K = \{0\} \times \mathbb{R}$ is a discrete subgroup of G . Let $S = \mathbb{R}$ (with the usual topology) and define a function $f : S \rightarrow G$ by the formula: $f(x) = (x, 0)$ for $x \in S$. It is easy to see that f is continuous, satisfies (1.1), and $f - (f + A_t) = A_t$ for every $t \in \mathbb{R}$, where $A_t : S \rightarrow K$ is defined by: $A_t(x) = (0, tx)$ for $x \in S$. Since $f - A_t$ is additive for every $t \in \mathbb{R}$, this completes the example.

Proposition 2.9. Assume that hypotheses (A_1) – (A_3) are valid, G is uniquely divisible by 2, and for some neighbourhood $U \subset G$ of 0 with $K \cap U = \{0\}$ we have

$$G = \bigcup \{2^n U : n \in \mathbb{N}\}.$$

Let $A_1, A_2 : S \rightarrow G$ be additive functions such that for some mapping $f : S \rightarrow G$ we have

$$f(x) - A_i(x) \in K \quad \text{for every } x \in S, \ i = 1, 2.$$

Then $A_1 = A_2$.

PROOF. Let $A(x) = A_1(x) - A_2(x)$ for $x \in S$. Then $A : S \rightarrow K$ is additive. Fix $x \in S$. Note that there is $n \in \mathbb{N}$ with $2^{-n}A(x) \in U$ and $2^{-n}A(x) = A(2^{-n}x) \in K$. Hence $2^{-n}A(x) = 0$ and consequently $A(x) = 0$. Thus we have shown that $A(x) = 0$ for every $x \in S$, which means that $A_1 = A_2$.

We end this part with a theorem concerning the Jensen difference.

Theorem 2.10. *Suppose (A_1) – (A_3) and (H_0) – (H_2) . Let $D \subset S$, $\text{int } D \neq \emptyset$ be a J -convex set and $h : D \rightarrow G$ be a function continuous at a point $x_0 \in \text{int } D$ satisfying*

$$(2.2) \quad 2h\left(\frac{x+y}{2}\right) - h(x) - h(y) \in K \quad \text{for every } x, y \in D.$$

Then there exists an additive function $A : S \rightarrow G$ and $y \in G$ with

$$(2.3) \quad h(x) - y - A(x) \in K \quad \text{for every } x \in D.$$

If, moreover, (H_3) holds, then A can be chosen, in a unique way, continuous at 0.

PROOF. It is easy to see that the function $f = p \circ h : S \rightarrow G/K$ is a solution of (1.4) continuous at x_0 . According to (H_1) there is a neighbourhood $W \subset S$ of 0 such that $W \subset V := \{y \in S : y + x_0 \in D\}$. Further, the J -convexity of D yields

$$\frac{x}{2} \in V \quad \text{for } x \in V$$

and, on account of (H_2) , $S = \bigcup \{2^n W : n \in \mathbb{N}\}$. Consequently V satisfies conditions (i) and (ii) of (H_2) . Thus, in view of Theorem 1.4, there are an additive function $g : S \rightarrow G/K$ and an element $u \in G/K$ both of them uniquely determined, such that (1.5) holds. It is easily seen that g is continuous at x_0 . Hence, by Theorem 2.2, there is an additive mapping $A : S \rightarrow G$ with $A(x) \in g(x)$ for $x \in S$. Moreover, if (H_3) holds, then A can be chosen continuous at 0. Since

$$h(x) - (A(x) + y) \in f(x) - (g(x) + u) \quad \text{for every } x \in D, y \in u,$$

we obtain (2.3).

Now suppose that (H_3) holds and $A_0 : S \rightarrow G$ is also an additive function continuous at 0 such that, for every $x \in D$, $h(x) - y_0 - A_0(x) \in K$ with some $y_0 \in G$. Then $f(x) = p(A_0(x)) + p(y_0)$ for $x \in D$. Thus Theorem 1.4 implies $p \circ A_0 = g$, because $p \circ A_0$ is additive. Hence, in view of Theorem 2.2, $A = A_0$. This ends the proof.

3. The Cauchy and Jensen congruences almost everywhere

In this part we consider the situation where the congruences (1.1) and (1.3) are valid not for every $x, y \in S$, but only almost everywhere in S^2 with respect to some ideal in S^2 . We start with some definitions.

Let $X \neq \emptyset$ be a set. We say that $J \subset 2^X$ is an ideal in X provided the following two conditions are valid:

$$\begin{aligned} &\text{if } A \in J \text{ and } B \subset A, \text{ then } B \in J; \\ &A \cup B \in J \quad \text{for every } A, B \in J. \end{aligned}$$

If the latter condition is replaced by the following:

$$\bigcup \{A_n : n \in \mathbb{N}\} \in J \quad \text{for every } \{A_n\}_{n \in \mathbb{N}} \subset J,$$

then J is called a σ -ideal. If, moreover, $X \notin J$, we say that J is a proper ideal (σ -ideal, respectively). Finally, if $(X, +)$ is a group, we say that J is a proper linearly invariant (abbreviated to p.l.i.) ideal (σ -ideal, resp.) provided J is a proper ideal (σ -ideal, resp.) in X and $\{x - A : x \in X\} \subset J$ for every $A \in J$.

Given a p.l.i. ideal J in a group $(X, +)$ we set

$$\Omega(J) = \{M \subset X^2 : \text{there is } A \in J \text{ with } M[x] \in J \text{ for every } x \in X \setminus A\},$$

where $M[x] = \{y \in X : (x, y) \in M\}$. It turns out that $\Omega(J)$ is a p.l.i. ideal in the product group $(X^2, +)$ (cf. [9], p. 220).

Next, let us recall the notion “almost everywhere”. Let $J_0 \subset X$ ($J_0 \subset X^2$, resp.) be an ideal in X (X^2 , resp.). We say that a property $P(x)$, $x \in D \subset X$ ($D \subset X^2$, resp.), holds J_0 -almost everywhere (abbreviated to J_0 -a.e.) in D provided there is a set $A \in J_0$ such that the property holds for every $x \in D \setminus A$.

In the sequel the following two hypotheses will be useful:

- (T₁) S is a subsemigroup of a group $(H, +)$ and $S - S = H$;
- (T₂) $J \subset 2^H$ is a p.l.i. ideal in H and $S \notin J$.

We will also need the following two theorems:

Theorem 3.1 (see [9], Theorem 1). *Suppose (T₁) and (T₂). Let $(Y, +)$ be a group (not necessarily commutative) and $f : S \rightarrow Y$ a function satisfying*

$$(3.1) \quad f(x + y) = f(x) + f(y) \quad \Omega(J)\text{-a.e. in } S^2.$$

Then there exists exactly one additive function $F : H \rightarrow Y$ such that $F(x) = f(x)$ J -a.e. in S .

Theorem 3.2 (see [12], Theorem 1). Assume (A_1) and (T_1) . Let $x_0 \in S$, $P = S - x_0$, $J \subset 2^H$ be a p.l.i. σ -ideal in H such that $S \notin J$ and

$$\frac{1}{2}A, 2A \in J \quad \text{for every } A \in J,$$

and let $(Y, +)$ be a group (not necessarily commutative), and $M \in \Omega(J)$. If $M^{-1} := \{(y, x) : (x, y) \in M\} \in \Omega(J)$ and $k : P \rightarrow Y$ is a function such that

$$k(x + y) = k(x) + k(y) \quad \text{for } (x, y) \in P^2 \setminus M \text{ with } x + y \in P,$$

then there is exactly one additive function $F : H \rightarrow Y$ with $F(x) = k(x)$ J -a.e. in P .

(Theorem 1 in [12] is proved under the assumption that H is commutative and uniquely divisible by 2. However, we do not assume this in our Theorem 3.2, because it results from (A_1) and (T_1) in a similar way as is shown in (1.7).)

In the sequel, given S and H satisfying hypotheses (H_0) and (T_1) , we consider ideals $J \subset 2^H$ such that

$$(3.2) \quad 0 \in \text{int } d(U \setminus A)$$

for every non-empty open set $U \subset S$ and every $A \in J$,

where $d(D) = \{x \in S : D \cap (x + D) \neq \emptyset\}$ for every $D \subset S$. For instance, if $H = S$ is a locally compact topological group and J is the σ -ideal of Haar zero subsets of S , or $H = S$ is an Abelian topological Polish group and J is the σ -ideal of Christensen zero subsets of S , then (3.2) holds (see e.g. [11] and [8], resp.). We have as well the following two propositions and remark.

Proposition 3.3. Let $(S, +)$ be a semigroup (not necessarily commutative) with zero and $J \subset 2^S$ be an ideal in S such that

$$x + A \in J \quad \text{for every } x \in S \text{ and } A \in J.$$

Suppose that (H_0) holds, for every $y \in S$ the translations $R_y, L_y : S \rightarrow S$ given by: $R_y(x) = x + y$ and $L_y(x) = y + x$ are continuous at 0, and for every neighbourhood $W \subset S$ of zero and every $y \in S$, $y + W \notin J$. Then (3.2) is valid.

PROOF. Let $U \subset S$ be a non-empty open set and $A \in J$. Put $U_0 = U \setminus A$ and fix $x \in U$. There is an open neighbourhood $V \subset S$ of zero with $V + x \subset U$. Fix $y \in V$ and note that $y + x \in U \cap (y + U)$ (because $x \in U$).

Let W_1 and W_2 be neighbourhoods of 0 in S such that $y + x + W_1 \subset U$ and $x + W_2 \subset U$. Then $y + x + (W_1 \cap W_2) \subset U \cap (y + U)$, which means that $U \cap (y + U) \notin J$. Since

$$(U \cap (y + U)) \setminus (U_0 \cap (y + U_0)) \subset A \cup (y + A) \in J,$$

we must have $U_0 \cap (y + U_0) \neq \emptyset$. In this way we have proved that $y \in d(U_0)$ for $y \in V$. This yields the statement.

Remark 3.4. Let $(S, +)$ be a group (not necessarily commutative) such that (H_0) , (H_1) hold and every open set is of the second category of Baire. Then the σ -ideal of the first category subsets of S satisfies the assumptions of Proposition 3.3, which means that (3.2) is valid. (See also [10], Theorem 2.)

Before we proceed to the next proposition let us remind that, given a topological space X , we say that a set is universally measurable provided, for every probability Borel measure m on X , there are Borel sets $B_1, B_2 \subset X$ with $m(B_1) = m(B_2)$ and $B_1 \subset A \subset B_2$ (cf. e.g. [6] and [7]). Further, a group $(H, +)$ is called semitopological if it is endowed with a topology such that the group operation is separately continuous with respect to either variable (cf. [10], p. 597).

We also need the following lemma, which seems to be quite known; however, for the convenience of the reader we prove it.

Lemma 3.5. *Suppose that X and Y are topological spaces and $A \subset Y$ is a universally measurable set. Let $h : X \rightarrow Y$ be a Borel mapping (i.e., for every Borel set $B \subset Y$, $h^{-1}(B)$ is a Borel set). Then $h^{-1}(A)$ is universally measurable.*

PROOF. Let m be a probability Borel measure on X . Put $m_h(B) = m(h^{-1}(B))$ for every Borel set $B \subset Y$. Then m_h is a probability Borel measure on Y . Thus there are Borel sets $B_1, B_2 \subset Y$ with $m_h(B_1) = m_h(B_2)$ and $B_1 \subset A \subset B_2$. Setting $D_1 = h^{-1}(B_1)$ and $D_2 = h^{-1}(B_2)$ we have $D_1 \subset h^{-1}(A) \subset D_2$ and $m(D_1) = m_h(B_1) = m_h(B_2) = m(D_2)$. This ends the proof.

Proposition 3.6. *Suppose that $(S, +)$ is a semitopological group with the topology generated by a complete metric, commutative and uniquely divisible by 2, the mapping $x \rightarrow 2x$ is continuous, the mappings $x \rightarrow \frac{x}{2}$ and $x \rightarrow -x$ are Borel, and (H_2) holds. Let*

$$J_0 = \{A \subset S : A \text{ is universally measurable and } 0 \notin \text{int } d(A)\}$$

and J be an ideal (σ -ideal, resp.) generated by J_0 . Then J is a p.l.i. ideal (σ -ideal, resp.) in S satisfying (3.2).

PROOF. By Lemma 3.5, $x + A, x - A \in J_0$ for every $A \in J_0, x \in S$ (note that in our case $d(D) = D - D$ for every $D \subset S$, because S is a group). Thus $x + A, x - A \in J$ for every $A \in J$ and $x \in S$. Hence, on account of Proposition 3.3, it remains to show that J does not contain any neighbourhood $W \subset S$ of 0.

For the proof by contradiction suppose that this is not the case. Then, by (H_2) ,

$$S = \bigcup \{2^n W : n \in \mathbb{N}\}$$

for some $W \in J$. Further, there is a sequence $\{V_i\}_{i \in \mathbb{N}} \subset J_0$ with $W \subset \bigcup \{V_i : i \in \mathbb{N}\}$. Thus

$$S = \bigcup \{2^n V_i : i, n \in \mathbb{N}\}.$$

Since, according to Lemma 3.5, for every $i, n \in \mathbb{N}$ the set $2^n V_i$ is universally measurable, we may use the following theorem of J. P. R. CHRISTENSEN.

Theorem 3.7 (see [6], Theorem 1; see also [7], p. 113). *Let $(S, +)$ be a commutative semigroup with the neutral element 0. Suppose that S is equipped with a topology generated by a complete metric such that all translations $T_a : x \rightarrow x + a$ are continuous. Let $\{A_i\}_{i \in \mathbb{N}} \subset 2^S$ be a denumerable covering of S . Then there is $k \in \mathbb{N}$ such that for every universally measurable set $U \supset A_k$ the set $d(U) = \{x \in S : U \cap (x + U) \neq \emptyset\}$ is a neighbourhood of 0.*

According to Theorem 3.7, there are $n, i \in \mathbb{N}$ such that $0 \in \text{int } d(2^n V_i)$. Next, $d(2^n V_i) = 2^n(V_i - V_i) = 2^n d(V_i)$. Hence, by the continuity of the mapping $x \rightarrow 2^n x$, we have $0 \in \text{int } d(V_i)$, a contradiction. This ends the proof.

Now, we are in a position to prove the following

Theorem 3.8. Suppose (A_1) – (A_3) , (T_1) , (T_2) , (H_0) – (H_2) , and (3.2). Let $f : S \rightarrow G$ be a function continuous at a point $x_0 \in S$ and fulfilling

$$f(x+y) - f(x) - f(y) \in K \quad \Omega(J)\text{-a.e. in } S^2.$$

Then there exists an additive function $A : S \rightarrow G$ such that

$$(3.3) \quad f(x) - A(x) \in K \quad J\text{-a.e. in } S.$$

If, moreover, (H_3) holds, then A can be chosen continuous at 0 in a unique way.

PROOF. Define a function $f_0 : S \rightarrow G/K$ by $f_0(x) = p(f(x))$ for $x \in S$. Then

$$f_0(x+y) = f_0(x) + f_0(y) \quad \Omega(J)\text{-a.e. in } S^2.$$

Thus, by virtue of Theorem 3.1, there exists exactly one additive function $F : H \rightarrow G/K$ such that $F(x) = f_0(x)$ J -a.e. in S . We will show that F is continuous at 0.

Fix a neighbourhood $U \subset G/K$ of 0. There is a neighbourhood $V \subset G/K$ of zero with $V - V \subset U$. Since f is continuous at 0, f_0 is continuous at 0, too. Hence there is a neighbourhood $W \subset S$ of 0 such that $f_0(W) \subset V$. Put $W_0 = \{x \in W : f_0(x) = F(x)\}$. Then $W \setminus W_0 \in J$. Hence, in view of (3.2), $0 \in \text{int } d(W_0)$. Moreover $d(W_0) \subset W_0 - W_0$, whence

$$F(d(W_0)) \subset F(W_0) - F(W_0) \subset V - V \subset U.$$

Thus we have proved that F is continuous at 0. Consequently, by Theorem 2.2, there exists an additive function $A : S \rightarrow G$ such that $A(x) \in F(x)$ for $x \in S$. Furthermore, if (H_3) is valid, then A can be chosen continuous at 0. It is easily seen that (3.3) holds.

To complete the proof suppose that hypothesis (H_3) is fulfilled and $A_0 : S \rightarrow G$ is also continuous at 0 and $f(x) - A_0(x) \in K$ J -a.e. in S . Put $F_0 = p \circ A_0$. It is easily seen that $F_0(x) = f_0(x)$ J -a.e. in S . Thus, according to Theorem 3.1, $F_0 = F|_S$. Hence $A(x), A_0(x) \in F(x)$ for $x \in S$, which, on account of Theorem 2.2, means that $A_0 = A$. This ends the proof.

In order to obtain for the Jensen difference a result analogous to Theorem 3.8 the following proposition is necessary.

Proposition 3.9. *Suppose (A_1) , (T_1) , and (T_2) . Let $(Y, +)$ be a group (not necessarily commutative), $M \in \Omega(J)$, and $h : S \rightarrow Y$ be a function such that*

$$(3.4) \quad 2h\left(\frac{x+y}{2}\right) = h(x) + h(y) \quad \text{for every } (x, y) \in S^2 \setminus M.$$

If one the following two conditions holds:

$$1^\circ \quad S = H;$$

$2^\circ \quad J$ is a σ -ideal, $\frac{1}{2}A, 2A \in J$ for every $A \in J$, and $M^{-1} \in \Omega(J)$, then there exists exactly one additive function $F : H \rightarrow Y$ and exactly one constant $u \in Y$ such that

$$(3.5) \quad h(x) = F(x) + u \quad J\text{-a.e. in } S.$$

The proof of Proposition 3.9 is very similar to the proof of Theorem 3 in [12]. However, since it is not identical and our assumptions are a little different, we present it.

PROOF of Proposition 3.9. According to the definition of $\Omega(J)$, there is $A_M \in J$ with $M[x] \in J$ for every $x \in S \setminus A_M$. Fix $x_0 \in S \setminus A_M$ and put $M_0 = (M[x_0] \times S) \cup (S \times M[x_0]) \cup M$. It is easy to see that $M_0 \in \Omega(J)$.

Take $(x, y) \in S^2 \setminus M_0$. Then $(x_0, x), (x_0, y) \in S^2 \setminus M$ and

$$\begin{aligned} 0 &= 2h\left(\frac{x+x_0}{2}\right) - h(x_0) - h(x), \\ 0 &= -h(y) - h(x_0) + 2h\left(\frac{x_0+y}{2}\right), \\ 2h\left(\frac{x+y}{2}\right) &= h(x) + h(y). \end{aligned}$$

Adding these equalities we get

$$2h\left(\frac{x+y}{2}\right) = 2h\left(\frac{x+x_0}{2}\right) - 2h(x_0) + 2h\left(\frac{x_0+y}{2}\right).$$

Now, replacing x by $x+x_0$ and y by $y+x_0$ we obtain

$$2h\left(\frac{x+y}{2} + x_0\right) = 2h\left(\frac{x}{2} + x_0\right) - 2h(x_0) + 2h\left(\frac{y}{2} + x_0\right)$$

for every $(x, y) \in ((S - x_0) \times (S - x_0)) \setminus (M_0 + (-x_0, -x_0))$. Hence the function $k : S - x_0 \rightarrow Y$, defined by

$$k(x) = 2h\left(\frac{x}{2} + x_0\right) - 2h(x_0) \quad \text{for every } x \in S - x_0$$

satisfies the condition

$$k(x+y) = k(x) + k(y) \text{ for } (x, y) \in (S - x_0)^2 \setminus (M_0 + (-x_0, -x_0)) \\ \text{with } x+y \in S - x_0.$$

Now we may use Theorems 3.1 and 3.2. On account of them there is an additive function $F : H \rightarrow Y$ and a set $A \in J$ such that

$$k(x) = F(x) \quad \text{for every } x \in (S - x_0) \setminus A.$$

It is easy to check that according to the definition of k

$$(3.6) \quad 2h\left(\frac{x}{2} + x_0\right) = F(x) + 2h(x_0) \quad \text{for every } x \in (S - x_0) \setminus A.$$

Observe that for every $x \in (S - x_0) \setminus (M[x_0] - x_0)$ we have $x + x_0 \in S$ and $(x_0, x + x_0) \notin M$. Thus (3.4) implies

$$2h\left(\frac{x}{2} + x_0\right) = h(x + x_0) + h(x_0) \quad \text{for } x \in (S - x_0) \setminus (M[x_0] - x_0).$$

Whence and by (3.6)

$$h(x + x_0) = F(x) + h(x_0) \quad \text{for } x \in (S - x_0) \setminus (A \cup (M[x_0] - x_0)).$$

Now, replacing $x + x_0$ by x we get

$$h(x) = F(x) - F(x_0) + h(x_0) \quad \text{for } x \in S \setminus ((A + x_0) \cup M[x_0]).$$

Since $(A + x_0) \cup M[x_0] \in J$, this yields (3.5).

The uniqueness of F and u results easily from (3.5) and Theorems 3.1 and 3.2. This ends the proof.

Finally we have the following

Theorem 3.10. *Suppose that (A_1) – (A_3) , (H_0) – (H_2) , (T_1) , (T_2) , and (3.2) are valid, $M \in \Omega(J)$, and one of conditions 1° , 2° of Proposition 3.9 holds. Let $h : S \rightarrow G$ be a function continuous at a point $x_0 \in S$ and such that*

$$2h\left(\frac{x+y}{2}\right) - h(x) - h(y) \in K \quad \text{for every } (x, y) \in S^2 \setminus M.$$

Then there exist an additive function $A : S \rightarrow G$ and a constant $y \in G$ with

$$(3.7) \quad h(x) - (A(x) + y) \in K \quad J\text{-a.e. in } S.$$

If, moreover, (H_3) holds, then, in a unique way, A can be chosen continuous at 0.

PROOF. Note that the function $h_0 = p \circ h : S \rightarrow G/K$ satisfies (3.4). Thus, on account of Proposition 3.9, there is an additive function $F : H \rightarrow G/K$ and a constant $u \in G/K$ such that (3.5) holds with $h = h_0$. Further, in the same way as in the proof of Theorem 3.8, one can show that $F|_S$ is continuous at 0. Hence, in view of Theorem 2.2, there is an additive function $A : S \rightarrow G$ with $A(x) \in F(x)$ for $x \in S$. Moreover, if (H_3) holds, then A can be chosen continuous at 0. It is easily seen that (3.7) holds with every $y \in u$.

To complete the proof suppose that (H_3) holds and $A_0 : S \rightarrow G$ is also an additive function continuous at 0 and such that

$$h(x) - (A_0(x) + y_0) \in K \quad J\text{-a.e. in } S$$

with some $y_0 \in G$. Define a function $F_0 : S \rightarrow G/K$ by

$$F_0(x) = p(A_0(x)) \quad \text{for } x \in S.$$

Then $F_0(x) + p(y_0) = h_0(x)$ J -a.e. in S . Thus, by Proposition 3.9, $F|_S = F_0$, which means that $A(x), A_0(x) \in F(x)$ for $x \in S$. Hence, in view of Theorem 2.2, $A = A_0$. This ends the proof.

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