On the valuations of complemented modular lattices of finite length

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By a *valuation* of a lattice L is meant a single-valued function v of L into itself which has the following property: for each pair x, y of elements of L

$$v(x)+v(y)=v(x\cap y)+v(x\cup y).$$

Let L be a lattice of finite length with the least element o and let $d(x)$ denote the length of the interval sublattice $[o, x]$. Then d is a single-valued function called the *height function* of L. If L is also modular, then by a well-known result of DEDEKIND ([1], p. 68) this function is a valuation of L.

In the theory of continuous geometries there is known ([2], p. 119) the following

Theorem 1. *Let L be a complemented modular simple lattice of finite length and d the height function of L. If v is any valuation of L, then there exist real constants $\varkappa$, λ such that*

$$v(x)=\varkappa d(x)+\lambda \tag{1}$$

for every element x of L.

In this note we give a proof of elementary character for this theorem.

By our assumptions, every element x of L may be represented as

$$x=p_1\cup\cdots\cup p_r \qquad (r \text{ finite})$$

where each p_j $(j=1,\ldots,r)$ is an atom of L ([1], p. 105). Without loss of generality we may assume the set $\{p_1\ldots,p_r\}$ to be independent ([1], p. 104); then

$$d(x)=r \tag{2}$$

and

$$\begin{aligned}v(x)&=v(p_1\cup\cdots\cup p_{r-1})+v(p_r)-v((p_1\cup\cdots\cup p_{r-1})\cap p_r)=\\&=v(p_1\cup\cdots\cup p_{r-1})+v(p_r)-v(o).\end{aligned}$$

The subset $\{p_1,\ldots,p_{r-1}\}$ is a fortiori independent. Therefore we get, by induction,

$$v(x)=v(p_1)+\cdots+v(p_r)+(r-1)\,v(o). \tag{3}$$

Since L is simple, to each pair p_j, p_k $(j,k=1,\dots,r;\, j\neq k)$ there exists a p_l (with $1\leqq l\leqq r$) such that

$$p_j\cup p_k=p_k\cup p_l=p_l\cup p_j$$

([1], pp. 120—121). It follows

$$\begin{aligned} v(p_j)&=v(p_j\cup p_l)+v(p_j\cap p_l)-v(p_l)=\\ &=v(p_j\cup p_l)+v(o)-v(p_l)=\\ &=v(p_k\cup p_l)+v(p_k\cap p_l)-v(p_l)=v(p_k)\end{aligned}$$

for each pair j, k; consequently

$$v(p_1)=\cdots=v(p_r)=\mu.$$

Thus (3) may be written as follows:

$$\begin{aligned} v(x)&=r\mu+(r-1)\,v(o)=\\ &=r\big(\mu+v(o)\big)-v(o).\end{aligned}$$

Taking $\mu+v(o)=\varkappa$ and $-v(o)=\lambda$, by (2) we get (1).

It is easy to see that if L is not simple, then the assertion of Theorem 1 does not remain valid. Consider, for example, the distributive lattice $D=\{o,a,b,i\}$ with the defining relations $o<a<i$, $o<b<i$, $a\cap b=o$, $a\cup b=i$ and define the function v by $v(o)=2$, $v(a)=6$, $v(b)=7$, $v(i)=11$. Then v is a valuation of D which obviously cannot be represented in the form (1).

In what follows we deal with complemented modular lattices of finite length which are not necessarily simple.

It is known ([1], pp. 120—121) that any complemented modular lattice L of finite length may be represented as a direct union

$$L=L_1\times\cdots\times L_n \qquad (n\text{ finite}) \tag{4}$$

of some simple lattices $L_1,\dots,L_n$. Obviously, each L_j $(j=1,\dots,n)$ is again a complemented modular lattice of finite length. Using Theorem 1, we prove the following

Theorem 2. *Let L be a complemented modular lattice of finite length having the direct decomposition* (4) *and let d_j $(j=1,\dots,n)$ denote the height function of L_j. If v is any valuation of L, then there exist real constants $\lambda_1,\dots,\lambda_n,\mu$ such that*

$$v(x)=\sum_{j=1}^{n}\lambda_j d_j(x_j)+\mu \tag{5}$$

for every element x of L, where x_j denotes the j-th component of x in the direct decomposition (4).

PROOF. Let o_j $(j=1, \dots, n)$ denote the least element of L_j. It is easy to see that the mapping v_j defined by

$$v_j(x_j) = v(o_1, \dots, o_{j-1}, x_j, o_{j+1}, \dots, o_n) \qquad (x_j \in L_j)$$

is a valuation of L_j. Therefore, by a straightforward calculation we get

$$\begin{aligned} v(x) = v(x_1, \dots, x_n) &= v\big((x_1, \dots, x_{n-1}, o_n) \cup (o_1, \dots, o_{n-1}, x_n)\big) = \\ &= v(x_1, \dots, x_{n-1}, o_n) + v(o_1, \dots, o_{n-1}, x_n) - v(o) = \\ &= v(x_1, \dots, x_{n-1}, o_n) + v_n(x_n) - v(o). \end{aligned}$$

Hence, by induction for n,

$$v(x) = \sum_{j=1}^{n} v_j(x_j) - (n-1)\, v(o). \tag{6}$$

But, by Theorem 1 there exist real constants λ_j, μ_j $(j=1, \dots, n)$ such that

$$v_j(x_j) = \lambda_j d_j(x_j) + \mu_j .$$

Thus (6) may be written in the form

$$v(x) = \sum_{j=1}^{n} \lambda_j d_j(x_j) + \sum_{j=1}^{n} \mu_j - (n-1)\, v(o).$$

Taking

$$\sum_{j=1}^{n} \mu_j - (n-1)\, v(o) = \mu,$$

we get (5). Hence our theorem is proved.

Bibliography

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