

Generalized power means for matrix functions II

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Abstract. In a recent paper, the authors obtained matrix versions of a number of inequalities involving generalized power means. Here these results are extended to corresponding inequalities for power means of several matrices.

1. Introduction

Generalized power means are defined by [1]:

$$(1) \quad M_{n,a}(x; w)_p = \begin{cases} \left(\frac{\sum_{i=1}^n w_i x_i^{a+p}}{\sum_{i=1}^n w_i x_i^p} \right)^{1/a}, & a \neq 0 \\ \exp \left\{ \frac{\sum_{i=1}^n w_i x_i^p \log x_i}{\sum_{i=1}^n w_i x_i^p} \right\}, & a = 0 \end{cases}$$

where $a, p \in \mathbb{R}$, $x, w \in \mathbb{R}_+^n$, $n \in \mathbb{N}$. Concerning inequalities for these means see [2–5].

Matrix version of such results are obtained in [6]. Here, we shall give analogous results for several matrices.

2. Preliminaries

Let $A \in C^{n \times n}$ be a normal matrix, i.e., $A^*A = AA^*$. Here A^* means $\bar{A}^t$, the transpose conjugate of A . There exists [7] a unitary matrix U such

that

$$(2) \quad A = U^*[\lambda_1, \lambda_2, \dots, \lambda_n]U$$

where $[\lambda_1, \dots, \lambda_n]$ is the diagonal matrix $(\lambda_j \delta_{ij})$, and where $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of A , each appearing as often as its multiplicity. A is Hermitian if and only if $\lambda_i, i \in I_n = \{1, 2, \dots, n\}$ are real. If A is Hermitian and all λ_i are strictly positive, then A is said to be positive definite. Assume now that $f(\lambda_i) \in C$, $i \in I_n$ is well defined. Then $f(A)$ may be defined by (see e.g. [7, p. 71] or [8, p. 90])

$$(3) \quad f(A) = U^*[f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)]U.$$

As before, if $f(\lambda_i)$, $i \in I_n$ are all real, then $f(A)$ is Hermitian. If, also, $f(\lambda_i) > 0$, $i \in I_n$, then $f(A)$ is positive definite.

We note that for the inner product

$$(4) \quad (f(A)x, x) = \sum_{i=1}^n |y_i|^2 f(\lambda_i)$$

where $y \in C^n$, $y = Ux$ and so $\sum_{i=1}^n |y_i|^2 = \sum_{i=1}^n |x_i|^2$.

If A is positive definite, so that $\lambda_i > 0$, $i \in I_n$ and $f(t) = t^r$ where $t > 0$ and $r \in \mathbb{R}$, we have $f(A) = A^r$.

3. Generalized power means for several matrices

Definition 1. Let A_j , $j = 1, \dots, k$ be positive definite Hermitian matrices; $x_j \in C^n$, $j = 1, \dots, k$; $a, p \in \mathbb{R}$, then the generalized power mean of A_j is defined by

$$(5) \quad M_k^p(A; x)_a = \left\{ \frac{\sum_{j=1}^k (A_j^{a+p} x_j, x_j)}{\sum_{j=1}^k (A_j^p x_j, x_j)} \right\}^{1/a}, \quad a \neq 0,$$

$$= \exp \left\{ \frac{\sum_{j=1}^k ((A_j^p \log A_j) x_j, x_j)}{\sum_{j=1}^k (A_j^p x_j, x_j)} \right\}, \quad a = 0.$$

First, we shall prove the following result:

Theorem 1. *Let a, b, p, q satisfy*

$$(6) \quad \left| |a| - |b| \right| + a + 2p \leq b + 2q.$$

Then for every positive definite Hermitian matrix $A_j, x_j \in C^n$, $j = 1, \dots, k$

$$(7) \quad M_k^p(A; x)_a \leq M_k^q(A; x)_b.$$

PROOF. Using (4) we have

$$(8) \quad M_k^p(A; x)_a = \begin{cases} \left(\frac{\sum_{j=1}^k \sum_{i=1}^n |y_{ji}|^2 \lambda_{ji}^{a+p}}{\sum_{j=1}^k \sum_{i=1}^n |y_{ji}|^2 \lambda_{ji}^p} \right)^{1/a}, & a \neq 0 \\ \exp \left\{ \frac{\sum_{j=1}^k \sum_{i=1}^n |y_{ji}|^2 \lambda_{ji}^p \log \lambda_{ji}}{\sum_{j=1}^k \sum_{i=1}^n |y_{ji}|^2 \lambda_{ji}^p} \right\}, & a = 0. \end{cases}$$

Now, Theorem 1 is a simple consequence of the following lemma ([2]).

Lemma 1. *Let $a, b, p, q \in \mathbb{R}$. Then*

$$(9) \quad M_{n,a}(x; w)_p \leq M_{n,b}(x; w)_q$$

holds for every $x \in \mathbb{R}^n$ ($x \neq 0$), if and only if (6) holds.

Similarly, using other results for generalized power means we can obtain (see also [6]).

Theorem 2. *Let $a, b_1, b_2, \dots, b_s, p, q_1, \dots, q_s \in \mathbb{R}$, $s \geq 2$. Further, let*

$$(10) \quad \begin{aligned} Q_0 &= a^- - p, & Q_i &= b_i^+ + q_i, & i &= 1, \dots, s \\ Q_0^* &= a^+ + p, & Q_i^* &= b_i^- - q_i, & i &= 1, \dots, s \end{aligned}$$

where $a^+ = (|a| + a)/2$ and $a^- = (|a| - a)/2$, and for $i = 0, 1, \dots, s$, let

$$(11) \quad H_i = \begin{cases} \left(\sum_{\substack{j=0 \\ j \neq i}}^s Q_j^{-1} \right)^{-1}, & \text{when } \prod_{\substack{j=0 \\ j \neq i}}^s Q_j \neq 0 \\ 0, & \text{when } \prod_{\substack{j=0 \\ j \neq i}}^s Q_j = 0. \end{cases}$$

Let A_j , $j = 1, \dots, k$, be normal matrices with eigenvalues in I ($I \subset C$); $f_i : I \rightarrow \mathbb{R}_+$ be strictly positive functions for $i = 1, \dots, s$; $x_j \in C^n$, $j = 1, \dots, k$. If $Q_i \geq 0$ and $H_i \geq Q_i^*$ ($i = 0, \dots, s$) then

$$(12) \quad M_k^p((f_1 \dots f_s)(A); x)_a \leq M_k^{q_1}(f_1(A); x)_{b_1} \dots M_k^{q_s}(f_s(A); x)_{b_s}.$$

Theorem 3. Let $a, b_1, \dots, b_s, p, q_1, \dots, q_s, A_j, x_j$ ($j = 1, \dots, k$), f_i ($i = 1, \dots, s$) be as in the previous theorem. If

$$(13) \quad \max\{p + a^+, 1\} \leq q_i + b_i^+; \quad \max\{p - a^-, 0\} \leq \min\{q_i - b_i^-, 1\}$$

hold for every $i = 1, \dots, s$, then

$$(14) \quad M_k^p((f_1 + \dots + f_s)(A); x)_a \leq M_k^{q_1}(f_1(A); x)_{b_1} + \dots + M_k^{q_s}(f_s(A); x)_{b_s}.$$

The reverse inequality in (14) holds if

$$(15) \quad \min\{p + a^+, 1\} \geq \max\{q_i + b_i^+, 0\}; \quad \min\{p - a^-, 0\} \geq q_i - b_i^-$$

is valid for $i = 1, \dots, s$.

Theorem 4. Let A_j , $j = 1, \dots, k$ be positive definite Hermitian matrices with eigenvalues λ_{ji} ($j = 1, \dots, k$; $i = 1, \dots, n$) such that $0 < m \leq \lambda_{ji} \leq M$. Then

$$(16) \quad M_k^q(A; x)_b \leq K(m, M)M^p(A; x)_a$$

where a, b, p, q are fixed numbers such that (6) holds and where $K(m, M)$ is defined by

$$K(m, M) = \Gamma_{b,q}(t_0, \gamma) / \Gamma_{a,p}(t_0, \gamma),$$

$\gamma = M/m$ and t_0 is the unique positive root of the equation

$$\lambda_{a,p}(\gamma)(\gamma^q + t)(\gamma^{b+q} + t) = \lambda_{b,q}(\gamma)(\gamma^p + t)(\gamma^{a+p} + t)$$

where, for $t > 0$,

$$(17) \quad \lambda_{a,p}(t) = \begin{cases} t^p \frac{t^{a-1}}{a}, & a \neq 0 \\ t^p \log t, & a = 0; \end{cases}$$

and

$$(18) \quad \Gamma_{a,p}(t, \gamma) = \begin{cases} ((\gamma^{a+p} + t)/(\gamma^p + t))^{1/a}, & a \neq 0 \\ \exp((\gamma^p \log \gamma)/(\gamma^p + t)), & a = 0. \end{cases}$$

4. Generalized quasi-arithmetic means for several matrices

Definition 2. Let $\phi : I \rightarrow \mathbb{R}_+$ ($I \subset \mathbb{R}$) be a strictly positive function, $F : I \rightarrow \mathbb{R}$ a strictly monotone function, A_j , $j = 1, \dots, k$, Hermitian matrices with eigenvalues in I , $x_j \in C^n$, $j = 1, \dots, k$. The generalized quasi-arithmetic mean of A_j is, for $x \neq 0$,

$$(19) \quad F_k(A; x, \phi) = F^{-1} \left\{ \frac{\sum_{j=1}^k ((\phi \cdot F)(A_j)x_j, x_j)}{\sum_{j=1}^k (\phi(A_j)x_j, x_j)} \right\}.$$

The following result is a consequence of Theorem 1 from [9, p. 262]:

Theorem 5. Let K, L, M be three differentiable strictly monotone functions from the closed interval I to $\mathbb{R}$; ϕ, ψ, χ , three functions from I to $\mathbb{R}_+$; $f : I^2 \rightarrow I$ such that for all $u, v, s, t \in I$ the following inequality is valid.

$$\begin{aligned} & \left(\frac{M \circ f(u, v) - M \circ f(t, s)}{M' \circ f(t, s)} \right) \frac{\chi \circ f(u, v)}{\chi \circ f(t, s)} \\ & \leq \left(\frac{K(u) - K(t)}{K'(t)} \right) \frac{\phi(u)}{\phi(t)} f_1(t, s) + \left(\frac{L(v) - L(s)}{L'(s)} \right) \frac{\psi(v)}{\psi(s)} f_2(t, s). \end{aligned}$$

Let A_j , $j = 1, \dots, k$ be normal matrices with eigenvalues in J and let $g, h : J \rightarrow I$ be given functions; $x_j \in C^n$, $j = 1, \dots, k$. Then

$$(20) \quad f(K_k(g(A); x, \phi), L_k(h(A); x, \psi)) \geq M_k(f(g(A), h(A)); x, \chi).$$

Theorem 6. With the notation of the previous theorem,

$$(21) \quad M_k(A; x, \chi) \leq K_k(A; x, \phi)$$

if, for all $u, t \in J$,

$$(22) \quad \frac{M(u) - M(t)}{M'(t)} \frac{\chi(u)}{\chi(t)} \leq \left(\frac{K(u) - K(t)}{K'(t)} \right) \frac{\phi(u)}{\phi(t)}.$$

PROOF. Immediate from the previous theorem taking $f(x, y) = x$, $g(x) = x$.

An important special case of Theorem 5 is when $f(x, y) = x + y$. Then we get

$$(23) \quad K_k(g(A); x, \phi) + L_k(h(A); x, \psi) \geq M_k(g(A) + h(A); x, \chi)$$

holds if for all u, v, s, t in J , we have

$$\frac{M(u+v) - M(t+s)}{M'(t+s)} \frac{\chi(u+v)}{\chi(t+s)} \leq \frac{K(u) - K(t)}{K'(t)} \frac{\phi(u)}{\phi(t)} + \frac{L(v) - L(s)}{L'(s)} \frac{\psi(v)}{\psi(s)}.$$

Generalized quasi-arithmetic means are not only extensions of generalized power means but are also generalizations of the quasi-arithmetic means. Therefore, in the next section, we give some other results for the quasi-arithmetic means for matrix functions.

5. The quasi-arithmetic means for several matrices

Definition 3. Let A_j , $j = 1, \dots, k$ be Hermitian matrices with eigenvalues $\lambda_{ji} \in J$ ($j = 1, \dots, k$; $i = 1, \dots, n$). Suppose that $F : J \rightarrow \mathbb{R}$ is a continuous and strictly monotone function. The quasi-arithmetic F -means is defined by

$$(24) \quad F_k(A; x) = F^{-1} \left\{ \sum_{j=1}^k (F(A_j)x_j, x_j) \right\}$$

where $x_j \in C^n$, $j = 1, \dots, k$ with $\sum_{j=1}^k (x_j, x_j) = 1$.

Theorem 7. Let F, G be two continuous functions with domain J , G increasing (decreasing). Then

$$(25) \quad F_k(A; x) \leq G_k(A; x)$$

holds if G is convex (concave) with respect to F , i.e., if the function $\phi(t) = (G \circ F^{-1})(t)$ is convex (concave). If G is decreasing (increasing) and G is convex (concave) with respect to F , inequality (25) is reversed.

This is a consequence of Theorem 4 from [9, pp. 226]. Similarly, we can use results for three means [9, pp. 246–253].

Let $K : [k_1, k_2] \rightarrow \mathbb{R}$, $L : [\ell_1, \ell_2] \rightarrow \mathbb{R}$, $M : [m_1, m_2] \rightarrow \mathbb{R}$, $f : [k_1, k_2] \times [\ell_1, \ell_2] \rightarrow [m_1, m_2]$; $g : J \rightarrow [k_1, k_2]$, $h : J \rightarrow [\ell_1, \ell_2]$ be given functions and let A_j be a Hermitian matrix with eigenvalues $\lambda_{ji} \in J$ ($j =$

$1, \dots, k; i = 1, \dots, n$). Here K, L and M are twice differentiable and strictly monotone functions, M is increasing. Consider the inequality

$$(26) \quad f(K_k(g(A); x), L_k(h(A); x)) \geq M_k(f(g(A), h(A)); x)$$

or its reverse.

Theorem 8. *Inequality (26) holds if the function $H(s, t) = M(f(K^{-1}(s), L^{-1}(t)))$ is concave. If H is convex, then inequality (26) is reversed.*

Remark. Theorem 8 is a generalization of Hölder's and Minkowski's inequalities. Thus, if $f(u, v) = u + v$, when $H(s, t) = M(K^{-1}(s) + L^{-1}(t))$, $E = K'/K''$, $T = L'/L''$, $S = M'/M''$ and all of $K', L', M', K'', L'', M''$ are positive, then (56) holds if $S(u + v) \geq E(u) + T(v)$. Moreover, if $f(u, v) = uv$ when $H(s, t) = M(K^{-1}(s)L^{-1}(t))$,

$$A(u) = \frac{K'(u)}{K'(u) + uK''(u)}, \quad B(u) = \frac{L'(u)}{L'(u) + uL''(u)},$$

$$C(u) = \frac{M'(u)}{M'(u) + uM''(u)}$$

and K', L', M', A, B, C are all positive, then (56) holds if $C(uv) \geq A(u) + B(v)$.

A special case of Theorem 8 is also [9, p. 253]

$$(27) \quad F_k\left(\frac{1}{2}(g(A) + h(A)); x\right) \leq \frac{1}{2}\{F_k(g(A); x) + F_k(h(A); x)\}$$

where the function F has continuous second derivatives and is strictly increasing and strictly convex and F'/F'' is concave.

We now give two converse inequalities. These inequalities can be obtained as consequences of results from [10]:

Theorem 9. *Let F and G be two strictly monotone continuous functions defined on J , G increasing (decreasing) and G convex (concave) with respect to F . Let A_j , $j = 1, \dots, k$ be Hermitian matrices with eigenvalues in $[m, M]$. Then*

$$(28) \quad (F(M) - F(m))G_k(A; x) - (G(M) - G(m))F_k(A; x) \\ \leq F(M)G(m) - G(M)F(m).$$

If G is decreasing (increasing) and G is convex (concave) with respect to F , inequality (28) is reversed.

Theorem 10. Let $\phi(u, v)$ be a real function defined on $J \times J$, non-decreasing in u , G increasing and convex with respect to F . Let A_j , $j = 1, \dots, k$ be Hermitian matrices with eigenvalues in $[m, M]$. Then

$$(29) \quad \phi(G_k(A; x), F_k(A; x)) \\ \leq \max_{\vartheta \in [0,1]} \phi[G^{-1}(\vartheta G(m) + (1 - \vartheta)G(M)), F^{-1}(\vartheta F(m) + (1 - \vartheta)F(M))].$$

References

- [1] J. ACZÉL and Z. DARÓCZY, Über verallgemeinerte quasilinear Mittelwerte, die mit Gewichts functionene gebildet sind, *Publ. Math. Debrecen* **10** (1963), 171–190.
- [2] Z. DARÓCZY and L. LOSONCZI, Über den Vergleich von Mittelwerten, *Publ. Math. Debrecen* **17** (1970), 289–297.
- [3] ZS. PÁLES, On Hölder-type inequalities, *J. Math. Anal. Appl.* **95** (1983), 457–466.
- [4] ZS. PÁLES, Generalization of the Minkowski Inequality, *Ibid.* **90** (1982), 456–462.
- [5] ZS. PÁLES, On complementary inequalities, *Publ. Math. Debrecen* **30** (1983), 75–88.
- [6] B. MOND and J. E. PEČARIĆ, Generalized power means for matrix functions, *Publ. Math. Debrecen* **46** (1995), 33–41.
- [7] M. MARCUS and H. MINC, A survey of matrix theory and inequalities, *Allyn and Bacon, Boston*, 1969.
- [8] R. BELLMAN, Introduction to Matrix Analysis, *McGraw-Hill, New York*, 1960.
- [9] P. S. BULLEN, D. S. MITRINOVIĆ and P. M. VASIĆ, Means and their inequalities, *D. Reidel, Dordrecht/Boston/Lancaster/Tokyo*, 1988.
- [10] J. E. PEČARIĆ and P. R. BEESACK, On Knopp's inequality for convex functions, *Canad. Math. Bull.* **30** (1987), 267–272.

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