

On the convergence of series of iterates

By M. A. McKIERNAN (Waterloo, Ontario)

Introduction

Let $f(x)$ be an analytic function of the complex argument x . Define the integral iterates $f^{[n]}(x)$ of $f(x)$ recursively by:

$$f^{[0]}(x) = x, \quad f^{[n+1]}(x) = f\{f^{[n]}(x)\} \quad \text{for } n = 0, 1, 2, \dots$$

Hence $f^{[n+1]}(x)$ is defined at x if and only if $f^{[n]}(x)$ is in the domain of f . It is well known ([1]) that functions $g(x)$ defined by series of the form

$$(1) \quad g(x) = \sum_{n=0}^{\infty} a_n \sum_{r=0}^n \binom{n}{r} (-\beta)^{n-r} \Phi\{f^{[r]}(x)\},$$

for analytic Φ , have applications to functional equations and related fields. If $\beta = 1$ and $f(x) = x + 1$, then (1) reduces to

$$g(x) = \sum_{n=0}^{\infty} a_n \Delta^n \Phi(x).$$

If $\beta = 1$ and $f(x) = \frac{x}{x+1}$ while $\Phi(x) = x$, then (1) becomes the factorial series

$$g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n n! a_n}{z(z+1)\dots(z+n)} \quad \text{where } z = \frac{1}{x}.$$

It was first noted by CAYLEY ([2]) and SCHRÖDER ([3]) that the series

$$(2) \quad \sum_{n=0}^{\infty} \frac{s(s-1)\dots(s-n+1)}{n!} \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} f^{[r]}(x),$$

when suitably convergent, converge to the generalized iterates $f^{[s]}(x)$, for arbitrary real or complex s . Similarly, by formally differentiating (2) with respect to s and evaluating at $s=0$, one would conjecture ([4]) that the series

$$(3) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} f^{[r]}(x),$$

when convergent, converges to a function $L(x)$ satisfying the functional differential

equation (for a particularly interesting application of this equation, see [5])

$$(4) \quad L\{f(x)\} = L(x) \cdot f'(x).$$

The convergence of the series $\sum a_n f^{[n]}(x)$, that is the case when $\beta = 0$ and $\Phi(x) = x$ in (1), was exhaustively studied by G. JULIA ([6]). The iterative properties of the sum function $f^{[s]}(x)$ of the series (2) were studied by C. BOURLET ([7]), essentially assuming convergence.

Let $\Phi^{[-1]}(x)$ denote the inverse function of the analytic function $\Phi(x)$. By substituting $\Phi^{[-1]}(x)$ into both sides of (1) and replacing $\Phi f \Phi^{[-1]}$ by a new f , one is led to consider the particular case

$$(5) \quad \sum_{n=0}^{\infty} a_n \sum_{r=0}^n \binom{n}{r} (-\beta)^{n-r} f^{[r]}(x),$$

since $(\Phi f \Phi^{[-1]})^{[r]} = \Phi f^{[r]} \Phi^{[-1]}$. The present paper develops some necessary and some sufficient conditions for the uniform convergence of (5) when $f(x)$ is analytic about x_0 , and $f(x_0) = x_0$. It will be assumed at first that $x_0 = 0$ since the general case is readily obtained from this case by a simple translation.

Preliminary theorems of iteration theory ([8])

Considerable use will be made of the following well known results from the theory of iteration.

Basic Theorem: *Given a function $f(x)$, analytic about $x=0$ and satisfying $f(0)=0$, $f'(0)=\alpha$ where $0 < |\alpha| < 1$. Then there exists a $\varrho > 0$ and a unique function $F(x)$, the Schröder function for $f(x)$, satisfying:*

(i) $F(x) = \lim_{n \rightarrow \infty} \alpha^{-n} \cdot f^{[n]}(x)$, the sequence converging uniformly in some neighbourhood of $x=0$;

(ii) $F(x)$ is analytic about $x=0$, $F(0)=0$ and $F'(0)=1$;

(iii) the inverse function, $F^{[-1]}(x)$, exists analytic about $x=0$ and $F^{[-1]}(x) = \sum_{n=1}^{\infty} c_n x^n$, uniformly convergent in $|x| \leq \varrho$, where $c_1 = 1$;

(iv) $F(x)$ satisfies the Schröder equation $F\{f^{[r]}(x)\} = \alpha^r \cdot F(x)$, whence

$$f^{[r]}(x) = \sum_{n=1}^{\infty} c_n \alpha^{rn} [F(x)]^n,$$

convergent for $|F(x)| \leq \varrho$ and all integer $r \geq 0$;

(v) the function $\Phi(x; z)$ defined by

$$\Phi(x; z) = \sum_{n=0}^{\infty} z^{-(n+1)} \cdot f^{[n]}(x)$$

is analytic in the entire complex plane except for simple poles at those $z = \alpha^n$, $n = 1, 2, \dots$ for which $c_n \neq 0$, and $z=0$ if these poles are not finite in number.

Since the coefficient $c_1 = 1 \neq 0$, the generating function $\Phi(x; z)$ always has a simple pole at $z = \alpha$. Further, since α^n approaches 0 with n when $0 < |\alpha| < 1$, if $\Phi(x; z)$ is analytic at $z = 0$ then $\Phi(x; z)$ must be a rational function of z , having only a finite number of poles in the extended plane. If $\Phi(x; z)$ is not a rational function of z , then $z = 0$ is an essential singularity, a limit point of simple poles.

With the series (5) we associate the series

$$(6) \quad h(z) = \sum_{n=0}^{\infty} a_n (z - \beta)^n, \text{ convergent say for } |z - \beta| < r.$$

It will be shown that the convergence or divergence of the series (5) is essentially determined by the behaviour of the series (6) at the poles of the generating function $\Phi(x; z)$.

For simplicity, throughout the remainder of this paper the symbols $h(z)$, $\Phi(x; z)$, $F(x)$, c_n , ϱ and r will always have the above meaning relative to $f(x)$, which is assumed to satisfy the hypotheses of the Basic Theorem.

By forming the convex hull of the singularities of $\Phi(x; z)$ it is clear that there will be at least one singularity z_0 of $\Phi(x; z)$ such that the distance from β to z_0 is not exceeded by the distance from β to any other singularity of $\Phi(x; z)$. Hence $z_0 = 0$ or $z_0 = \alpha^k$ for some k such that $c_k \neq 0$, and $|z_0 - \beta| \cong |\alpha^n - \beta|$ for all n for which $c_n \neq 0$. It has already been shown [9] that if $|z_0 - \beta| < r$, then (5) converges. The present paper extends this result, and includes the case $|z_0 - \beta| = r$ and $|z_0 - \beta| > r$.

Principal results

Since the proofs of theorems 1 and 2 below are lengthy. They will be presented after the proof of theorem 4.

Theorem 1. *A sufficient condition that (5) converge uniformly in x , for $|F(x)| \leq \varrho$, is that the series (6) converge uniformly on the set of singularities of $\Phi(x; z)$. If $z = 0$ is a limit point of singularities of $\Phi(x; z)$, and if (6) converges in $|z - \beta| < |\beta|$, whence $z = 0$ is on the circle of convergence of (6), then a sufficient condition for (5) to converge uniformly in $|F(x)| \leq \varrho |\alpha|^{v+1}$ is that*

(i) *for some $v > 0$, $N^{-v} \left| \sum_{n=1}^N a_n \beta^n \right|$ be bounded in N , and that*

(ii) *the singularities of $\Phi(x; z)$, except $z = 0$, lie interior to the intersection of the region $|z - \beta| < |\beta|$ and the angular sector defined by*

$$\arg \beta - \pi/2 + \varepsilon \leq \arg z \leq \arg \beta + \pi/2 - \varepsilon$$

for some $\varepsilon > 0$.

PROOF. (postponed).

Theorem 2. *Assume that there exists a K such that $c_K \neq 0$ whence α^K is a simple pole of the generating function, and that for some $0 < \theta < 1$,*

$$\theta |\alpha^K - \beta| \leq |z - \beta| \text{ for any singularity } z \neq \alpha^K \text{ of } \Phi(x; z).$$

If α^K lies outside the circle of convergence of (6), then the series (5) diverges for all x satisfying $|F(x)| \leq \varrho$, with the exception of those x for which $F(x) = 0$.

PROOF. (postponed).

Corollary 1. The condition that $n^{(1-v)} |a_n \beta^n|$ be bounded in n for some $v \neq 0$, or the condition that $\Sigma a_n \beta^n$ be Cesaro summable (C, v) for some $v > 0$, implies the condition (i) of theorem 1.

PROOF. That summability (C, v) implies condition (i) is well known [10], as is the first condition since, if bounded by P ,

$$\begin{aligned} \left| N^{-v} \sum_{n=1}^N a_n \beta^n \right| &\leq N^{-v} \sum_{n=1}^N n^{(1-v)} |a_n \beta^n| n^{v-1} \leq P N^{-v} \left\{ 1 + \int_1^N x^{v-1} dx \right\} = \\ &= \frac{P}{v} \{1 + (v-1)N^{-v}\}, \quad \text{q. e. d.} \end{aligned}$$

Theorem 3. For real α , $0 < \alpha < 1$, the Cayley—Schröder series (2) converges uniformly in $|F(x)| \leq q\alpha$ if $s \geq 0$, and uniformly in $|F(x)| \leq q\alpha^{-s+1}$ if $s < 0$.

PROOF. Since a Newton series $\Sigma \binom{s}{n} a_n$ converges in a half line $s > s_0$, it suffices to prove the theorem for $s = -\bar{s}$ where $\bar{s} > 0$. In terms of $\bar{s}$ the Cayley—Schröder series becomes

$$\sum_{n=0}^{\infty} (-1)^n \binom{\bar{s}-1+n}{n} \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} f^{[r]}(x),$$

and since $\sum_{n=0}^{\infty} (-1)^n \binom{\bar{s}-1+n}{n}$ is summable [11] $(C, \bar{s})$ for $\bar{s} > 0$, it is sufficient by corollary 1 to choose $v = \bar{s}$. Finally, since $0 < \alpha < 1$, it follows that all α^n are in the interval $[0, \alpha]$, and since $\beta = 1$, the poles of $\Phi(x; z)$ must lie in the angular sector specified in theorem 1, q. e. d.

Theorem 4. For real $-1 < \alpha < 0$, if $f(x)$ satisfies the further property that $f(-x) = -f(x)$, then the continuation ([9]) of the Cayley—Schröder series

$$(7) \quad e^{i\pi s} \sum_{n=0}^{\infty} \binom{s}{n} (-1)^n \sum_{r=0}^n \binom{n}{r} (+1)^{n-r} f^{[r]}(x)$$

converges uniformly in $|F(x)| \leq q|\alpha|$ for $s \geq 0$, and uniformly in $|F(x)| \leq q|\alpha|^{-s+1}$ for $s < 0$.

PROOF. Again it is sufficient to assume $s < 0$, so let $s = -\bar{s}$ where $\bar{s} \geq 0$, and the series in (7) becomes

$$e^{-i\pi\bar{s}} \sum_{n=0}^{\infty} \frac{\Gamma(\bar{s}+n)}{n\Gamma(\bar{s})\Gamma(n)} \sum_{r=0}^n \binom{n}{r} (+1)^{n-r} f^{[r]}(x).$$

But since [11]

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\bar{s}-1}} \frac{\Gamma(n+\bar{s})}{n\Gamma(n)} = \lim_{n \rightarrow \infty} \frac{\Gamma(n+\bar{s})}{n^{\bar{s}}\Gamma(n)} = 1$$

it follows again by corollary 1 that v may be chosen $v = \bar{s}$. It remains to show that the poles of $\Phi(x; z)$ lie in the appropriate angular sector, where in this case $\beta = -1$. But since $f(x)$ is odd, by induction we have

$$f^{[n+1]}(-x) = f^{[n]}\{-f(x)\} = -f^{[n]}\{f(x)\} = -f^{[n+1]}(x),$$

whence all $f^{[n]}(x)$ are odd, and by the Basic Theorem (i) it follows that $F(x)$, and hence $F^{[-1]}(x)$, is also odd. Hence $c_n = 0$ when n is even, while $-1 < \alpha < 0$ implies $-1 < \alpha^n < 0$ when n is odd, that is, when $c_n \neq 0$. Hence the poles of $\Phi(x; z)$ lie in the proper angular sector, q. e. d.

It should be noted that the series (7) also represents the generalized iterates $f^{[s]}(x)$ since

$$x^s = \{(-1) + (x+1)\}^s = e^{i\pi s} \{1 - (x+1)\}^s = e^{i\pi s} \sum_{n=0}^{\infty} \binom{s}{n} (-1)^n (x+1)^n.$$

While the iterative character of these series can be shown directly from the series themselves, as in [7], it will be easier to show these properties after presenting the relations required for the proofs of theorems 1 and 2. At that time the convergence of the series (3), and its relation to the functional differential equation (4) will be discussed.

PROOF OF THEOREMS 1 AND 2:

Lemma 1. *If $f(x)$ satisfies the hypotheses of the basic theorem, then*

$$(12) \quad \sum_{r=0}^n \binom{n}{r} (-\beta)^{n-r} f^{[r]}(x) = \sum_{m=1}^{\infty} c_m (\alpha^m - \beta)^n [F(x)]^m,$$

and hence

$$(13) \quad \sum_{n=0}^N a_n \sum_{r=0}^n \binom{n}{r} (-\beta)^{n-r} f^{[r]}(x) = \sum_{m=1}^{\infty} c_m \left\{ \sum_{n=0}^N a_n (\alpha^m - \beta)^n \right\} [F(x)]^m,$$

for all $|F(x)| \leq \varrho$.

PROOF. Both (12) and (13) are finite sums of the expansion of the Basic Theorem (iv) which is convergent for $|F(x)| \leq \varrho$, q. e. d.

In view of (13), the proof of the first part of theorem 1 is clear, for if the partial sums $\sum_{n=0}^N a_n (\alpha^m - \beta)^n$ are uniformly bounded for all N and all m for which $c_m \neq 0$, then (13) can be expected to converge. Similarly if the series (6) diverges at some α^K for which $c_K \neq 0$, then at least one term in the series (13) becomes unbounded and the series can be expected to diverge. We now proceed with the details.

Specifically set

$$(14) \quad G(M, N, x) = \sum_{m=1}^M c_m \left\{ \sum_{n=0}^N a_n (\alpha^m - \beta)^n \right\} [F(x)]^m$$

or equivalently

$$(14') \quad G(M, N, x) = \sum_{m=1}^M c_m (\alpha^m)^{v+1} \left\{ \sum_{n=0}^N a_n (\alpha^m - \beta)^n \right\} [\alpha^{-(v+1)} F(x)]^m.$$

If now (6) converges uniformly at all α^m for which $c_m \neq 0$, then the partial sums $\sum a_n(\alpha^m - \beta)^n$ are uniformly bounded at these α^m , so that there exists a $P > 0$ such that

$$\left| \sum_{n=0}^N a_n(\alpha^m - \beta)^n \right| \leq P \text{ for all } N \text{ and all } m, \text{ for which } c_m \neq 0.$$

Applying this to (14), it follows that, for $|F(x)| \leq \varrho$,

$$\left| \sum_{m=1}^M c_m \left\{ \sum_{n=0}^N a_n(\alpha^m - \beta)^n \right\} [F(x)]^m \right| \leq \sum_{m=1}^{\infty} |c_m| \varrho^m \cdot P,$$

which converges. By the Weierstrass M test it follows that $\lim_{M \rightarrow \infty} G(M, N, x)$ converges uniformly in N and in x , for $|F(x)| \leq \varrho$. Further, since (6) converges at the α^m for which $c_m \neq 0$, it follows that $\lim_{N \rightarrow \infty} G(M, N, x)$ exists for all M , uniformly in $|F(x)| \leq \varrho$.

Hence

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n \sum_{r=0}^n \binom{n}{r} (-\beta)^{n-r} f^{[r]}(x) &= \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} G(M, N, x) = \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} G(M, N, x) = \\ &= \lim_{M \rightarrow \infty} \sum_{m=1}^M c_m h(\alpha^m) [F(x)]^m, \end{aligned}$$

which converges uniformly in $|F(x)| \leq \varrho$ since $h(\alpha^m)$ is bounded when $c_m \neq 0$. This proves the first part of theorem 1. To prove the second part of theorem 1, we need:

Lemma 2. *Given an analytic function $h(z)$ whose expansion $h(z) = \sum_{n=1}^{\infty} a_n(z - \beta)^n$ about $z = \beta$ converges for $|z - \beta| < |\beta|$. If*

$$(15) \quad N^{-v} \sum_{r=1}^N a_r \beta^r, \text{ for some } v > 0,$$

is bounded for all N , then $z^{v+1}h(z)$ and $z^{v+1} \sum_{r=1}^N a_r(z - \beta)^r$ are uniformly bounded for all N and all z within the intersection of some neighbourhood of $z = 0$ and the angular sector

$$(16) \quad \arg \beta + \pi/2 - \varepsilon \leq \arg z \leq \arg \beta - \pi/2 + \varepsilon$$

for some $\varepsilon > 0$.

PROOF. Let P denote an upper bound of (15), then using ABEL's identity

$$\begin{aligned} \left| z^{v+1} \sum_{n=1}^N a_n \beta^n \left(\frac{z - \beta}{\beta} \right)^n \right| &= |z|^{v+1} \left| \sum_{n=1}^{N-1} \left\{ \frac{\sum_{r=1}^n a_r \beta^r}{n^v} \right\} n^v \left\{ \left(\frac{z - \beta}{\beta} \right)^n - \left(\frac{z - \beta}{\beta} \right)^{n+1} \right\} + \right. \\ &\quad \left. + N^v \left(\frac{z - \beta}{\beta} \right)^N \frac{\sum_{r=1}^N a_r \beta^r}{N^v} \right| \leq \\ &\leq P \cdot |z|^{v+1} \left\{ \left| 1 - \frac{z - \beta}{\beta} \right| \sum_{n=1}^{N-1} n^v \left| \frac{z - \beta}{\beta} \right|^n + N^v \left| \frac{z - \beta}{\beta} \right|^N \right\}, \end{aligned}$$

and for z within the circle of convergence $|z - \beta| < |\beta|$, by setting $z = \beta(1 - y)$ or equivalently $y = \frac{\beta - z}{\beta}$, it follows that $|y| < 1$ and that the inequality may be written

$$(17) \quad \leq 2P|\beta|^{v+1} \left\{ \frac{|1-y|}{1-|y|} \right\}^{v+1} (1-|y|)^{v+1} \sum_{n=1}^{\infty} n^v |y|^n.$$

But as a particular case of PRINGSHEIM's theorem (for example see [11], p. 180.) it follows that

$$\lim_{|y| \uparrow 1} (1-|y|)^{v+1} \sum_{n=1}^{\infty} n^v |y|^n$$

exists, bounded, and hence since

$$\frac{|1-y|}{1-|y|}$$

is bounded (for example [10], p. 438.) in the angular sector $-\pi/2 + \varepsilon \leq \arg(1-y) \leq \pi/2 - \varepsilon$, it follows that (17) is uniformly bounded for all y in this sector and in some neighbourhood of $y=1$. But since $z = \beta(1-y)$, y in this region implies z satisfies (16). Since the partial sums are uniformly bounded, clearly $z^{v+1}h(z)$ is also bounded, q. e. d.

If, as is assumed in the hypotheses of theorem 1, those α^n for which $c_n \neq 0$ lie in the region described in lemma 2, and if (15) holds, then

$$(\alpha_n)^{v+1} \sum_{r=1}^N a_r (\alpha^n - \beta)^r$$

is uniformly bounded, say by P_1 , for all N and all n for which $c_n \neq 0$. Hence, using (14'), we have that the terms of $G(M, N, x)$ are bounded by

$$\sum_{m=1}^M |c_m| P_1 |\alpha^{-(v+1)} F(x)|^m \leq P_1 \sum_{m=1}^M |c_m| \varrho^m$$

when $|F(x)| \leq \varrho |\alpha|^{v+1}$. Hence using the Weierstrass M -test, it follows that $\lim_{M \rightarrow \infty} G(M, N, x)$ exists, uniformly in N and in x for $|F(x)| \leq \varrho |\alpha|^{v+1}$. Further, for

any given M , all the α^n , $n \leq M$, for which $c_n \neq 0$ lie inside the circle of convergence of $h(z)$, so that $\lim_{N \rightarrow \infty} G(M, N, x)$ exists for every M , uniformly for $|F(x)| \leq \varrho |\alpha|^{v+1}$.

Hence as before

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=0}^N a_n \sum_{r=0}^n \binom{n}{r} (-\beta)^{n-r} f^{[r]}(x) &= \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} G(M, N, x) = \lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} G(M, N, x) = \\ &= \lim_{M \rightarrow \infty} \sum_{m=1}^M c_m (\alpha^m)^{v+1} h(\alpha^m) \cdot [\alpha^{-v-1} F(x)]^m. \end{aligned}$$

But again by lemma 2, $(\alpha^m)^{v+1} h(\alpha^m)$ is bounded, say by P_2 and for $|F(x)| \leq \varrho |\alpha|^{v+1}$, this last series is majorized by $\sum_{m=1}^{\infty} |c_m| \varrho^m P_2$ which converges. This proves theorem 1.

To prove theorem 2, we need the

Lemma 3. (General KOENIG's Theorem). *Let $f(x)$ satisfy the hypothesis of the Basic Theorem. Assume that there exists a K such that $c_K \neq 0$ and that, for some $0 < \theta < 1$,*

$$\theta |\alpha^K - \beta| \cong |\alpha^n - \beta| \text{ whenever } c_n \neq 0.$$

Let

$$g_n(x) = \sum_{s=0}^n \binom{n}{s} (-\beta)^{n-s} f^{[s]}(x).$$

Then

$$\lim_{n \rightarrow \infty} \frac{g_n(x)}{(\alpha^K - \beta)^n} = c_K [F(x)]^K$$

uniformly in $|F(x)| \cong \varrho$.

PROOF. It follows from (12) that for $|F(x)| \cong \varrho$,

$$\left| \frac{g_n(x)}{(\alpha^K - \beta)^n} - c_K [F(x)]^K \right| = \left| \sum_{\substack{m=1 \\ m \neq K}}^{\infty} c_m [F(x)]^m \left\{ \frac{\alpha^m - \beta}{\alpha^K - \beta} \right\}^n \right| \cong \theta^n \sum_{m=1}^{\infty} |c_m| \varrho^n,$$

and since $0 < \theta < 1$, the lemma follows, q. e. d.

In the particular case that $\beta = 0$, then $g_n(x) = f^{[n]}(x)$ and, since $0 < |\alpha| < 1$ it follows that $\theta |\alpha| \cong |\alpha^n|$ for all $n > 1$, for $\theta = |\alpha|$, whence we may choose $K = 1$ and obtain KOENIG's theorem, that is, Basic Theorem (i).

Lemma 4. *Under the hypothesis of lemma 3,*

$$\lim_{n \rightarrow \infty} |g_n(x)|^{1/n} = |\alpha^K - \beta|$$

for all x satisfying $|F(x)| \cong \varrho$ and $F(x) \neq 0$.

PROOF. Since $c_K \neq 0$, and $F(x) \neq 0$ it follows that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln \frac{|g_n(x)|}{|\alpha^K - \beta|^n} = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |c_K [F(x)]^K| = 0$$

from which the lemma follows, q. e. d.

We may now prove theorem 2 as follows. Since the radius of convergence of $h(z) = \sum a_n(z - \beta)^n$ is r , it follows that

$$\overline{\lim}_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \frac{1}{r}.$$

In view of lemma 4, it follows that, for $|F(x)| \cong \varrho$ and $F(x) \neq 0$,

$$\overline{\lim}_{n \rightarrow \infty} |a_n g_n(x)|^{1/n} = \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n} \cdot \lim_{n \rightarrow \infty} |g_n(x)|^{1/n} = \frac{|\alpha^K - \beta|}{r}.$$

Hence by the Cauchy root test, if $|\alpha^K - \beta| > r$, the series $\sum a_n g_n(x)$, that is the series (5), diverges for all x satisfying $|F(x)| \cong \varrho$ and $F(x) \neq 0$. This completes the proof of theorem 2.

Properties of the series

Theorem 5. Let $f(x)$ be analytic about $0=f(0)$, and $f'(0)=\alpha$ for real $0<\alpha<1$. Then the series

$$(3) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} f^{[r]}(x)$$

converges uniformly in $|F(x)| \leq \varrho \alpha^{1+\varepsilon}$, for any $\varepsilon > 0$, to an analytic function $L(x)$, where

$$(18) \quad L(x) = \ln \alpha \cdot \frac{F(x)}{F'(x)},$$

and $L(x)$ satisfies the equation

$$(19) \quad L\{f(x)\} = L(x) \cdot f'(x).$$

PROOF. As in theorem 3, the singularities in z of the generating function $\Phi(x; z)$ all lie in the interval $[0, \alpha]$ on the real axis. Since in this case $\beta=1$, and since the series $\sum \frac{(-1)^n}{n}$ converges, the v of theorem 1 may be chosen $v = \varepsilon > 0$. Hence (3) converges uniformly in $|F(x)| \leq \varrho \alpha^{1+\varepsilon}$ to an analytic function $L(x)$. But by (12), each term of this series can be expanded as a power series in $F(x)$ each convergent for $|F(x)| \leq \varrho$, and hence also for $|F(x)| \leq \varrho \alpha^{1+\varepsilon}$. Hence by the Weierstrass double series theorem, the order of summation may be interchanged, which yields

$$L(x) = \sum_{m=1}^{\infty} c_m \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} (\alpha^n - \beta)^n \right\} [F(x)]^m = -\ln \alpha \cdot \sum_{m=1}^{\infty} m c_m [F(x)]^m.$$

But since

$$x = f^{[0]}(x) = \sum_{m=1}^{\infty} c_m [F(x)]^m,$$

differentiation term by term shows that (18) holds. Finally, since $F\{f(x)\} = \alpha F(x)$ while $F'\{f(x)\} \cdot f'(x) = \alpha F'(x)$, clearly (19) follows, q. e. d.

Theorem 6. If $f(x)$ is analytic about $0=f(0)$, and $f'(0)=\alpha$ for real $0<\alpha<1$, then the functions $f^{[s]}(x)$ defined by

$$(2) \quad f^{[s]}(x) = \sum_{n=0}^{\infty} \binom{s}{n} \sum_{r=0}^n \binom{n}{r} (-1)^{n-r} f^{[r]}(x),$$

the series being uniformly convergent in $|F(x)| \leq \varrho \alpha^{-s+1}$ if $s < 0$, $|F(x)| \leq \varrho \alpha$ if $s > 0$, satisfy the relations

$$(i) \quad F\{f^{[s]}(x)\} = \alpha^s F(x)$$

$$(ii) \quad f^{[s]} \{ f^{[r]}(x) \} = f^{[s+r]}(x),$$

where clearly if s is a positive integer, (2) reduces to the integral iterate of $f(x)$.

PROOF. It is sufficient to prove (i) since $F^{[-1]}(x)$ exists. As in the proof of theorem 5, each term of (2) can be expanded into the power series in $F(x)$ given in

(12), and the order of summation interchanged for x within the region of uniform convergence of (2). But then we obtain

$$f^{[s]}(x) = \sum_{m=1}^{\infty} c_m \left\{ \sum_{n=0}^{\infty} \binom{s}{n} (\alpha^m - 1)^n \right\} [F(x)]^m,$$

and since $0 < \alpha < 1$, this becomes, in view of Basic Theorem (iii),

$$f^{[s]}(x) = \sum_{m=1}^{\infty} c_m [\alpha^s F(x)]^m = F^{[-1]} \{ \alpha^s F(x) \}, \quad \text{q. e. d.}$$

Clearly, as in theorem 4, similar theorems hold for $\beta = -1$ and for $f'(0) = \alpha$ where $-1 < \alpha < 0$, provided $f(-x) = -f(x)$.

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