On the explicit form of n-group operations

By M. HOSSZÚ (Miskolc)

An n-group is a non-empty set G in which an operation $F(x_1, \ldots, x_n)(n \geqq 2)$ is defined with the following properties:

(I) $x_i \to F(x_1, \ldots, x_n)$ *maps G onto the whole of G for any one i of the indices* $1, \ldots, n$ *and for arbitrary constants* x_k $(k \neq i)$ *in G (transitivity);*

(II) $F[F(x_1, x_2, \ldots, x_n), x_{n+1}, \ldots, x_{2n-1}] =$

$$= F[x_1, x_2, \ldots, x_i, F(x_{i+1}, x_{i+2}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}]$$

holds for every $i = 1, \ldots, n-1$ *and for each* $x_k \in G$ *(associativity).*

The 2-groups are the usual binary groups.

For the previous results on n-groups we refer to [2—15]. E. L. Post [8] has proved that every n-group G has a binary covering group K in which the n-group operation has the form

$$F(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n, \qquad x_i \in G \subseteqq K.$$

However, here the group operation xy is defined on K and $z = xy$ does not lie necessarily in G for every x and y in G. Here we give an explicit form of n-group operations by means of a binary group operation defined on the *same* set G and by a certain automorphism of this binary group. The following theorem will be proved:

Theorem. *An operation F defined on a set G is an n-group operation if and only if it has the form*

$$F(x_1, \ldots, x_n) = x_1 x_2^{\alpha} x_3^{\alpha^2} \ldots x_n^{\alpha^{n-1}} c \tag{1}$$

where xy is a binary group operation defined on the same set G and $x \to x^{\alpha}$ *is an automorphism of this binary group the* $n-1$*-th power of which is an inner automorphism:*

$$x^{\alpha^{n-1}} = cxc^{-1}, \tag{2}$$

further $c \in G$ *is a fixed element unchanged by the automorphism* α*:*

$$c^{\alpha} = c. \tag{3}$$

It can be verified immediately that any function of the form (1) with an automorphism α satisfying (2), (3) is an n-group operation. Thus in order to prove the Theorem we must show only that every n-group operation F can be written in the form (1), where α is an automorphism with properties (2), (3). This will be proved by proving a sequence of lemmas.

Lemma 1. *If $c_2, \ldots, c_{n-1} \in G$ are fixed, then $xy \overset{\text{def}}{=} F(x, c_2, \ldots, c_{n-1}, y)$ is a binary group operation on G.*

This is evident by (I)—(II) if we put $i = n-1$, $x_{n+j} = x_{j+1} = c_{j+1}$, $(j = 1, \ldots, n-2)$.

Lemma 1 implies that for fixed x_k $(k \neq i)$

$$x_i \to F(x_1, x_2, \ldots, x_n), \qquad i = 1, n$$

are 1—1 mappings of G onto the whole of G. We show that a similar statement holds for $i = 2, \ldots, n-1$.

Lemma 2. *For fixed $c_2, \ldots, c_{n-1} \in G$*

$$x \to x^{\eta_i} \overset{\text{def}}{=} F(c_i, \ldots, c_{n-1}, x, c_2, \ldots, c_i)$$

are 1—1 mappings of G onto G.

This is clear since

$$y \to (xy)z = F\big[F(x, c_2, \ldots, y), c_2, \ldots, z\big] =$$
$$= F\big[x, c_2, \ldots, c_{i-1}, F(c_i, \ldots, c_{n-1}, y, c_2, \ldots, c_i), c_{i+1}, \ldots, z\big] = \varphi(y^{\eta_i})$$

is a 1—1 mapping[1]) for every fixed $x, z \in G$, further $y \to y^{\eta_i}$ maps G onto G on account of (I).

Lemma 3. *Every n-group operation F defined on G has the form*

$$F(x_1, \ldots, x_n) = x_1 x_2^{\varepsilon_2} \ldots x_{n-1}^{\varepsilon_{n-1}} x_n, \tag{4}$$

where xy is a binary group operation defined in Lemma 1, further, $x \to x^{\varepsilon_i}$ is the inverse mapping of $x \to x^{\eta_i}$ defined in Lemma 2.

The proof is evident by (II) if we take

$$F(x_1, x_2^{\eta_2}, \ldots, x_{n-1}^{\eta_{n-1}}, x_n) =$$
$$= F[x_1, F(c_2, \ldots, x_2, c_2), F(c_3, \ldots, x_3, c_2, c_3), \ldots, F(c_{n-1}, x_{n-1}, \ldots), x_n] =$$
$$= F\{\ldots F[F(x_1, c_2, \ldots, x_2), c_2, c_3, \ldots, x_3], c_2, \ldots, x_n\} = \{\ldots[(x_1 x_2) x_3] \ldots\} x_n$$

into account.

Lemma 4. *A function F of the form* (4) *satisfies (II) only if we have*

$$x^{\varepsilon_i} = a_i x^{\alpha_i}, \qquad (i = 2, \ldots, n-1), \tag{5}$$

where $a_i \in G$ is a constant and $x \to x^{\alpha_i}$ is an automorphism.

This can be proved by putting (4) into (II):

$$(x_1 x_2^{\varepsilon_2} \ldots x_{n-1}^{\varepsilon_{n-1}} x_n) x_{n+1}^{\varepsilon_2} x_{n+2}^{\varepsilon_3} \ldots x_{2n-1} = \tag{6}$$
$$= x_1 x_2^{\varepsilon_2} \ldots x_{i-1}^{\varepsilon_{i-1}} (x_i x_{i+1}^{\varepsilon_2} \ldots x_{i+j-1}^{\varepsilon_j} \ldots x_{i+n-2}^{\varepsilon_{n-1}} x_{i+n-1})^{\varepsilon_i} x_{i+n}^{\varepsilon_{i+1}} \ldots x_{2n-1}.$$

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[1]) The composition $\varphi[\psi(x)]$ of one valued functions φ and ψ is 1-1 only if ψ is 1-1. This is clear since $\psi(x) = \psi(y)$ implies $\varphi[\psi(x)] = \varphi[\psi(y)]$.

Let e denote the unit element of the binary group defined on G in Lemma 1. Then choosing

$$x_{i+1}^{\varepsilon_2} = \ldots = x_{i+n-2}^{\varepsilon_{n-1}} = e$$

we get

$$(x_i x_{i+n-1})^{\varepsilon_i} = x_i^{\varepsilon_i} a_i^{-1} x_{i+n-1}^{\varepsilon_i}$$

showing that

$$x^{\alpha_i} \underline{\underline{\mathrm{def}}}\, a_i^{-1} x^{\varepsilon_i}$$

is an automorphism.[2])

Lemma 5. *The automorphisms α_i defined above satisfy the functional equations:*

$$(7) \qquad x^{\alpha_j \alpha_i} \underline{\underline{\mathrm{def}}}\, (x^{\alpha_j})^{\alpha_i} = c_{j,i} x^{\alpha_k} c_{j,i}^{-1}, \qquad k \equiv j + i - 1 \pmod{n-1},$$

where $c_{j,i} \in G$ are constants and $x^{\alpha_n} = x^{\alpha_1} = x$.

This is a consequence of (6) if we take (5) into account and keep x_k $(k \neq i+j-1)$ fixed[3]):

$$(x_{i+j-1}^{\alpha_j})^{\alpha_i} = \begin{cases} c_{j,i} x_{i+j-1}^{\alpha_{i+j-1-(n-1)}} d_{j,i}, & \text{if} \quad i+j>n; \\ c_{j,i}^* x_{i+j-1}^{\alpha_{i+j-1}} d_{j,i}^*, & \text{if} \quad i+j \leqq n. \end{cases}$$

Since the automorphisms α_i leave the unit element e fixed, by putting $x_{i+j-1} = e$ here we obtain $d_{j,i} = c_{j,i}^{-1}$.

Lemma 6. *Each of the automorphisms α_i satisfying (7) can be expressed by means of a power e.g. of the automorphism α_2:*

$$x^{\alpha_i} = b_i x^{\alpha_2^{i-1}} b_i, \qquad (i = 3, 4, \ldots, n).$$

This is true according to the formulae

$$x^{\alpha_2^2} = c_{2,2} x^{\alpha_3} c_{2,2}^{-1},$$

$$x^{\alpha_2^3} = (x\alpha_2^2)\alpha_2 = c_{2,2}^{\alpha_2} x^{\alpha_3 \alpha_2} (c_{2,2}^{-1})^{\alpha_2} = c_{2,2}^{\alpha_2} c_{3,2} x^{\alpha_4} c_{3,2}^{-1} (c_{2,2}^{\alpha_2})^{-1}$$

obtained by the repeated application of (7).

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[2]) Here we have

$$a_i^{-1} = x_{i+1}^{\varepsilon_{i+1}} x_{i+2}^{\varepsilon_{i+2}} \ldots x_n x_{n+1}^{\varepsilon_2} \ldots x_{n+i-2}^{\varepsilon_{i-1}},$$

which is a constant depending only on the choice of $c_2, \ldots, c_{n-1}$ since here $x_{i+1}, \ldots, x_{i+n-2}$ are fixed.

[3]) Here we have

$$c_{j,i} = [a_i(x_i x_{i+1}^{\varepsilon_i} \ldots x_{i+j-2}^{\varepsilon_{j-1}} a_j)^{\alpha_i}]^{-1} x_i^{\varepsilon_i} \ldots x_{i+j-2}^{\varepsilon_{i+j-2-(n-1)}} a_{i+j-1-(n-1)},$$

$$d_{j,i} = x_{i+j}^{\varepsilon_{i+j-(n-1)}} \ldots x_{i+n-1}^{\varepsilon_i} [(x_{i+j}^{\varepsilon_{j+1}} \ldots x_{i+n-1})^{\alpha_i}]^{-1},$$

$$c_{j,i}^* = [a_i(x_i x_{i+1}^{\varepsilon_2} \ldots x_{i+j-2}^{\varepsilon_{j-1}} a_j)^{\alpha_i}]^{-1} x_i^{\varepsilon_i} \ldots x_{i+j-2}^{\varepsilon_{i+j-2}} a_{i+j-1},$$

$$d_{j,i}^* = x_{i+j}^{\varepsilon_{i+j}} \ldots x_{i+n-1}^{\varepsilon_i} [(x_{i+j}^{\varepsilon_{j+1}} \ldots x_{i+n-1})^{\alpha_i}]^{-1}.$$

On account of lemmas 3,4 and 6 we get the form

$$F(x_1, \dots, x_n) = x_1 b_2 x_2^{\alpha_2} b_3 x_3^{\alpha_2^2} b_4 x_4^{\alpha_2^3} \dots x_{n-1}^{\alpha_2^{n-2}} b_n x_n,$$

(8) $$F(x_1, \dots, x_n) = x_1 x_2^{\alpha} a_3 x_3^{\alpha^2} a_4 \dots a_n x_n$$

where $x^{\alpha} \overset{\text{def}}{=} b_2 x^{\alpha_2} b_2^{-1}$ is an automorphism and $a_3, \dots, a_n$ are constants.

Lemma 7. *The F of the form* (8) *satisfies* (*II*) *only if*

(9) $$(a_3 \dots a_n)^{\alpha} = a_3 a_4 \dots a_n,$$

(10) $$a_k x = x a_k, \qquad x \in G, (k = 3, \dots, n-1)$$

hold.

Namely, in the case $i = 2$ we have by (II) and (8):

$$(x_1 x_2^{\alpha} a_3 x_3^{\alpha^2} \dots a_n x_n) x_{n+1}^{\alpha} a_3 x_{n+2}^{\alpha^2} \dots a_n x_{2n-1} =$$

$$= x_1 (x_2 x_3^{\alpha} a_3 x_4^{\alpha^2} \dots a_n x_{n+1})^{\alpha} a_3 x_{n+2}^{\alpha^2} \dots a_n x_{2n-1}.$$

If we put here $x_k = e$, it becomes (9). If we put $x_k = e$ for every $k \neq 3$, then taking also (9) into account, (II) for $i = 4$ gives

$$a_3 x_3^{\alpha^2} a_4 \dots a_n = x_3^{\alpha^2} (a_3 \dots a_n)^{\alpha} = x_3^{\alpha^2} a_3 \dots a_n,$$

that is

$$a_3 y = y a_3$$

holds for every $y = x_3^{\alpha^2} \in G$. Therefore, if we put $x_k = e$ for every $k \neq 4$, then by a similar reasoning we obtain

$$a_3 a_4 x_4^{\alpha^3} a_5 \dots a_n = a_3^{\alpha} x_4^{\alpha^3} (a_4 a_5 \dots a_n)^{\alpha}.$$

Being a_3 commutable for every element of G, a_3^{α} is also, hence

$$a_3^{\alpha} x_4^{\alpha^3} (a_4 a_5 \dots a_n)^{\alpha} = x_4^{\alpha^3} a_3^{\alpha} (a_4 a_5 \dots a_n)^{\alpha} =$$

$$= x_4^{\alpha^3} (a_3 a_4 \dots a_n)^{\alpha} = x_4^{\alpha^3} a_3 a_4 \dots a_n = a_3 x_4^{\alpha^3} a_4 \dots a_n,$$

i. e.

$$a_4 y = y a_4$$

is true for any $y = x_4^{\alpha^3} \in G$.

Thus (10) can be proved for every $k < n$.

By the Lemma 7 (8) can be written as follows:

(11) $$F(x_1, x_2, \dots, x_n) = x_1 x_2^{\alpha} x_3^{\alpha^2} \dots x_{n-1}^{\alpha^{n-2}} c x_n,$$

where $c \overset{\text{def}}{=} a_3 \dots a_n$ satisfies

$$c^{\alpha} = c$$

in accordance with (9).

A simple calculation shows that (11) satisfies (II) ($i = 1$) only if

2) $$x^{\alpha^{n-1}} = c x c^{-1}$$

holds. Taking (2) into acount, (11) can be written in the more symmetric form

$$F(x_1, \dots, x_n) = x_1 x_2^{\alpha} x_3^{\alpha^2} \dots x_{n-1}^{\alpha^{n-2}} c x_n c^{-1} c = x_1 x_2^{\alpha} x_3^{\alpha^2} \dots x_n^{\alpha^{n-1}} c. \tag{1}$$

This completes the proof of the Theorem.

Since every continuous group defined on a real interval is isomorphic to the real additive group and every continuous automorphism of a real additive group is a homogeneous linear function [1], in the special case, where $I = G$ is a real interval and $F(x_1, \dots, x_n)$ is continuous, the following corollary is an easy consequence of our Theorem:

Corollary. *The most general form of continuous n-group operations* $F(x_1, \dots, x_n)$ *on a real interval* I *is*

$$F(x_1, \dots, x_n) = f^{-1}\left[\sum_{i=1}^{4} \alpha^{i-1} f(x_i)\right], \qquad x_i \in I,$$

where $f(x)$ *is a continuous and strictly monotonic function the inverse function of which is* f^{-1} *and* α *is a constant for which*

$$\alpha^{n-1} = 1, \text{ i. e. } \alpha = \begin{cases} 1 & \text{if } n \text{ is even;} \\ \pm 1 & \text{if } n \text{ is odd} \end{cases}$$

holds. [14—15]

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