Rings on a direct sum of cyclic groups

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The purpose of this note is to give necessary and sufficient conditions that two rings on the same direct sum of cyclic groups be isomorphic. The problem is mentioned in [3], page 265, but is not discussed there because of the intricacy of the subject. In this paper we simplify the complications of the problem by an appropriate choice of matrix notation.

Let G and H be two rings (which may or may not be associative) such that:

$$G^+ = H^+ = \sum_{\alpha < S} \oplus \{u_\alpha\}$$

where S is an ordinal number and the direct summands $\{u_\alpha\}$ are cyclic groups. Since a finite cyclic group is the direct sum of cyclic groups of prime power order, we may assume that there exist ordinal numbers, S_{ij}, where $i, j = 1, 2, 3, \ldots$, $S_{ij} \leqq S$ for all i, j, $S_{ij} \leqq S_{mn}$ if and only if either $i < m$ or $i = m$ and $j \leqq n$, and $\alpha \leqq S_{ij}$ for some i, j for every ordinal $\alpha < S$, such that:

1) $\{u_\alpha\}$ is infinite cyclic if $\alpha < S_{11}$

2) $\{u_\alpha\}$ is cyclic of order p_i^j if $S_{ij} \leqq \alpha < S_{i\,j+1}$ where $p_1, p_2, p_3, \ldots$ is the sequence of rational prime numbers in natural order.

We now consider matrices with well-ordered row and column vectors whose entries are rational integers and having at most a finite number of non-zero entries in each row. Such a matrix whose column vectors have order type r and whose row vectors have order type s will be called an *r by s row-finite matrix*. The matrix product of an r by s row-finite matrix $(a_{\alpha\beta})$ and an s by t row-finite matrix $(b_{\alpha\beta})$ can be defined in the usual way to be the r by t row-finite matrix $(a_{\alpha\beta})(b_{\alpha\beta}) = (c_{\alpha\beta})$, where $c_{\alpha\beta} = \sum_{\nu} a_{\alpha\nu} b_{\nu\beta}$. If $(d_{\alpha\beta})$ is a u by v row-finite matrix, then the left Kronecker product of $(a_{\alpha\beta})$ and $(d_{\alpha\beta})$ can also be defined in the usual way (using the arithmetic of ordinal numbers given in [4]) to be the ur by vs row-finite matrix $(a_{\alpha\beta}) \otimes (d_{\alpha\beta}) = (e_{\alpha\beta})$, where $e_{\alpha\beta} = a_{\nu\varepsilon} d_{\sigma\tau}$, $\alpha = u\nu + \sigma$ and $\beta = v\varepsilon + \tau$.

Returning now to the ordinal number S described in the second paragraph above, we make the following definitions:

Definition 1. If $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ are two r by S row-finite matrices such that:

1) $a_{\alpha\beta} = b_{\alpha\beta}$ whenever $\beta < S_{11}$

2) $a_{\alpha\beta} \equiv b_{\alpha\beta} \pmod{p_i^j}$ whenever $S_{ij} \leqq \beta < S_{i,j+1}$

then the matrices $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ will be called *S-equivalent* and we will write

$$(a_{\alpha\beta}) \overline{\overline{S}} (b_{\alpha\beta}).$$

Definition 2. An S by S row-finite matrix $(a_{\alpha\beta})$ will be called an *S-matrix* if the following are satisfied:

1) If $\alpha < S_{11}$ then $a_{\alpha\beta} \neq 0$ for some $\beta < S_{11}$

2) If $S_{ij} \leqq \alpha < S_{i,j+1}$ then:

a) $a_{\alpha\beta} = 0$ whenever $\beta < S_{11}$.

b) $a_{\alpha\beta} \equiv 0 \pmod{p_m^n}$ whenever $m \neq i$ and $S_{mn} \leqq \beta < S_{m,n+1}$.

c) $a_{\alpha\beta} \equiv 0 \pmod{p_i^{n-j}}$ whenever $n \geqq j$ and $S_{in} \leqq \beta < S_{i,n+1}$.

d) $a_{\alpha\beta} \not\equiv 0 \pmod{p_i^{n-j+1}}$ for some $n \geqq j$ and some β, where $S_{in} \leqq \beta < S_{i,n+1}$.

Definition 3. An S-matrix, A, will be called *S-invertible* if there exists an S-matrix, B, such that:

$$AB \overline{\overline{S}} BA \overline{\overline{S}} (\delta_{\alpha\beta})$$

where $\delta_{\alpha\alpha} = 1$ for all α, and $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$.

By the theorem of BEAUMONT (see [2], or [3], page 264) multiplication in the ring G will be completely determined by a set of multiplication coefficients $\{g_{\alpha\beta\nu}\}$, defined by $u_\alpha u_\beta = \sum_\nu g_{\alpha\beta\nu} u_\nu$, where the integers $g_{\alpha\beta\nu}$ satisfy:

1) $g_{\alpha\beta\nu} = 0$ for all but a finite number of values of ν.

2) $g_{\alpha\beta\nu} = 0$ if $\nu < S_{11}$ and either $S_{11} \leqq \alpha$ or $S_{11} \leqq \beta$.

3) $g_{\alpha\beta\nu} \equiv 0 \pmod{p_i^j}$ if $S_{ij} \leqq \nu < S_{i,j+1}$

and either $S_{mn} \leqq \alpha < S_{m,n+1}$, $m \neq$ i or $S_{mn} \leqq \beta < S_{m,n+1}$, $m \neq$ i.

4) $g_{\alpha\beta\nu} \equiv 0 \pmod{p_i^{j-k}}$ if $S_{ij} \leqq \nu < S_{i,j+1}$, $S_{im} \leqq \alpha \prec S_{i,m+1}$, $S_{in} \leqq \beta < S_{i,n+1}$, $k = \min[m, n]$, and $k < j$.

A fifth condition on the numbers $g_{\alpha\beta\nu}$ can be given which is equivalent to associativity in G (see [3], page 264, or [2]).

Our final simplification is as follows:

Definition 4. If $\{g_{\alpha\beta\nu}\}$ is a set of multiplication coefficients for a ring G satisfying the conditions above, then the matrix $(g_{\alpha\beta})$, where $g_{\alpha\beta} = g_{\nu\varepsilon\beta}$, $\alpha = S\nu + \varepsilon$ will be called a *G-matrix*.

Thus a G-matrix will be an S^2 by S row-finite matrix and any two G-matrices are S-equivalent.

We may now prove our result, which is an extension of BEAUMONT's result on the isomorphism of algebras over a ring (see [1], Theorem 5).

Theorem. *Let G and H be two rings satisfying the conditions described in the second paragraph of this paper. If $(g_{\alpha\beta})$ is a G-matrix and $(h_{\alpha\beta})$ is an H-matrix, then G and H are isomorphic if and only if there exists an S-invertible S-matrix A such that*

$$(g_{\alpha\beta}) A \overline{\overline{S}} (A \otimes A)(h_{\alpha\beta}).$$

PROOF. Let f be a mapping from G to H. Then f will be additive if and only if it is determined uniquely by $fu_\alpha = \sum_{\beta < S} a_{\alpha\beta} u_\beta$ for each $\alpha < S$, where the matrix $A = (a_{\alpha\beta})$ is some S by S row-finite matrix. The mapping f will be order-preserving if and only if A is an S-matrix, and will be 1 to 1 and onto if and only if A is S-invertible.

Finally

$$f(u_\alpha u_\beta) = f(\sum_{\nu<S} g_{\alpha\beta\nu} u_\nu) = \sum_{\substack{\nu<S \\ \eta<S}} g_{\alpha\beta\nu} a_{\nu\eta} u_\eta$$

and

$$(fu_\alpha)(fu_\beta) = (\sum_{\nu<S} a_{\alpha\nu} u_\nu)(\sum_{\nu<S} a_{\beta\nu} u_\nu) = \sum_{\substack{\nu<S \\ \zeta<S \\ \eta<S}} a_{\alpha\nu} a_{\beta\zeta} h_{\nu\zeta\eta} u_\eta .$$

so that f will be multiplicative if and only if

$$\sum_{\nu<S} g_{\alpha\beta\nu} a_{\nu\eta} u_\eta = \sum_{\substack{\nu<S \\ \zeta<S}} a_{\alpha\nu} a_{\beta\zeta} h_{\nu\zeta\eta} u_\eta \quad \text{for each} \quad \alpha<S,\ \beta<S,$$

and $\eta<S$. These equalities hold if and only if the matrices $(g_{\alpha\beta})A$ and $(A\otimes A)(h_{\alpha\beta})$ are S-equivalent.

Bibliography

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