

Common fixed points of compatible set-valued mappings

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Abstract. In a recent paper IMDAD and AHMAD [1] proved several fixed point theorems for set-valued mappings satisfying conditions weaker than commuting, such as weakly commuting, quasi-commuting, and slightly commuting. In this paper we show that two of these definitions are special cases of δ -compatibility, and prove a fixed point theorem for four maps satisfying the δ -compatibility condition.

Let (X, d) be a complete metric space, $B(X)$ the collection of all nonempty bounded subsets of X . For $A, B \in B(X)$, define $\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}$. Let F be a multivalued map from X into $B(X)$, I a single-valued selfmap of X . From [4] and [2], the pair (F, I) is weakly commuting on X if for any x in X ,

$$\delta(FIx, IFx) \leq \max \{\delta(Ix, Fx), \text{diam } IFx\}$$

quasi-commuting on X if, for any x in X $IFx \subseteq FIx$,

slightly commuting on X if for any x in X ,

$$\delta(FIx, IFx) \leq \max \{\delta(Ix, Fx), \text{diam } Fx\}.$$

From [3] the pair (F, I) is δ -compatible if $\delta(IFx_n, FIx_n) \rightarrow 0$ whenever $\{x_n\}$ is a sequence in x such that $Ix_n \rightarrow t$ and $Fx_n \rightarrow \{t\}$ for some $t \in X$. As noted in [3], weakly commuting and slightly commuting imply δ -compatibility.

Let Φ be the set of all real-valued functions $\phi : (\mathbb{R}^+)^5 \rightarrow \mathbb{R}^+$ which are semi-continuous from the right and nondecreasing in each of the coordinate variables such that $\phi(t, t, t, at, bt) < t$ for each $t > 0$, $a, b \geq 0$, $a + b \leq 4$.

Let Ψ denote the set of real-valued functions $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which are upper semi-continuous from the right and nondecreasing with $\psi(t) < t$ for $t > 0$.

Theorem. *Let $F, G : X \rightarrow B(X)$, I, J two selfmaps of X . Suppose that (F, I) and (G, J) are δ -compatible, one of the four maps is continuous, $F(X) \subseteq J(X)$, $G(X) \subseteq I(X)$, and $\phi \in \Phi$. Suppose that*

$$(1) \quad \begin{aligned} & \delta(Fx, Gy) \\ & \leq \phi(\delta(Ix, Fx), \delta(Jy, Gy), \delta(Ix, Gy), \delta(Jy, Fx), d(Ix, Jy)) \end{aligned}$$

$\psi \in \Psi$, satisfies, for $t > 0$, $a, b > 0$, $a + b \leq 4$,

$$\psi(t) = \max \left\{ \begin{array}{l} \phi(t, t, t, at, bt), \quad \phi(t, 0, 0, t, 0), \quad \phi(0, 0, t, t, t), \\ \phi(0, t, t, 0, 0) \end{array} \right\} < t.$$

Then F, G, I and J have a unique common fixed point z such that $Iz = Jz = z$ and $Fz = Gz = \{z\}$. Also, z is the unique common fixed point of F and I , and of G and J ; i.e., $z = Iz = Jz = Fz = Gz$.

PROOF. Let $x_0 \in X$, y_1 an arbitrarily chosen point in $X_1 := Fx_0$. Since $F(X) \subseteq J(X)$, there exists a point $x_1 \in X$ such that $Jx_1 = y_1$. Choose an arbitrary point y_2 in $X_2 := Gx_1$. Since $G(X) \subseteq I(X)$, there exists a point $x_2 \in X$ with $Ix_2 = y_2$. In general, for any $x_{2n} \in X$ with $y_{2n+1} \in X_{2n+1} := Fx_{2n}$, there exists an $x_{2n+1} \in X$ satisfying $Jx_{2n+1} = y_{2n+1}$. For $y_{2n+2} \in X_{2n+2} := Gx_{2n+1}$, there exists an $x_{2n+2} \in X$ such that $Ix_{2n+2} = y_{2n+2}$. Set $V_n := \delta(X_n, X_{n+1})$.

Case 1. Suppose there exists an n for which $V_n = 0$. Then, if n is even we have $\delta(Gx_{2n-1}, Fx_{2n}) = 0$. This implies that $Fx_{2n} = y_{2n+1} = Jx_{2n+1} = Gx_{2n+1} = y_{2n+2} = Ix_{2n+2}$. Since G and J are δ -compatible, from Proposition 3.1 of [3],

$$(2) \quad GJx_{2n+1} = JGx_{2n+1} = CGx_{2n+1}.$$

From (1),

$$\begin{aligned} \delta(Fx_{2n+2}, Gx_{2n+1}) & \leq \phi(\delta(Fx_{2n+2}, Gx_{2n+1}), 0, 0, \delta(Fx_{2n+2}, Gx_{2n+1}), 0) \\ & \leq \psi(\delta(Fx_{2n+2}, Gx_{2n+1}), 0, 0, \delta(Fx_{2n+2}, Gx_{2n+1}), 0) \\ & < \delta(Fx_{2n+2}, Gx_{2n+1}) \end{aligned}$$

which implies that $Fx_{2n+2} = Gx_{2n+1}$. From the δ -compatibility of F and I ,

$$(3) \quad IFx_{2n+2} = FIFx_{2n+2} = FFx_{2n+2}.$$

Again from (1),

$$\begin{aligned}
\delta(FFx_{2n+2}, Fx_{2n+2}) &= \delta(FFx_{2n+2}, Gx_{2n+1}) \\
&\leq \phi(0, 0, \delta(FFx_{2n+2}, Fx_{2n+2}), \delta(FFx_{2n+2}, Fx_{2n+2}), \\
&\quad \delta(FFx_{2n+2}, Fx_{2n+2})) \\
&\leq \psi(0, 0, \delta(FFx_{2n+2}, Fx_{2n+2}), \delta(FFx_{2n+2}, Fx_{2n+2}), \\
&\quad \delta(FFx_{2n+2}, Fx_{2n+2})) \\
&< \delta(FFx_{2n+2}, Fx_{2n+2}),
\end{aligned}$$

and $FFx_{2n+2} = Fx_{2n+2}$. Therefore Fx_{2n+2} is a fixed point of F . Using (3), Fx_{2n+2} is a fixed point of I . Since $Fx_{2n+2} = Gx_{2n+1}$, using (1),

$$\begin{aligned}
\delta(Gx_{2n+1}, GGx_{2n+1}) &= \delta(Gx_{2n+2}, GGx_{2n+1}) \\
&\leq \phi(0, 0, \delta(Gx_{2n+1}, GGx_{2n+1}), \delta(Gx_{2n+1}, GGx_{2n+1}), \\
&\quad \delta(Gx_{2n+1}, GGx_{2n+1})) \\
&\leq \psi(0, 0, \delta(Gx_{2n+1}, GGx_{2n+1}), \delta(Gx_{2n+1}, GGx_{2n+1}), \\
&\quad \delta(Gx_{2n+1}, GGx_{2n+1}))
\end{aligned}$$

which yields that $Gx_{2n+1} = GGx_{2n+1}$, and hence Gx_{2n+1} is a fixed point of G and, from (2), a fixed point of J . Thus Fx_{2n+2} is a common fixed point of F, G, I , and J .

Case 2. Suppose that $V_n > 0$ for all n . Then, using the same argument as in [2], it follows that $\{y_n\}$ is a Cauchy sequence and hence converges to a point z in X . Thus $\lim y_{2n} = \lim Ix_{2n} = \lim y_{2n+1} = \lim Jx_{2n+1} = z$ and $\lim Fx_{2n} = \lim Gx_{2n+1} = \{z\}$. From Proposition 3.1 of [3], $\lim \delta(FIx_{2n}, IFx_{2n}) = 0$.

Suppose that I is continuous. Then $\lim Iy_{2n} = Iz$, and, from the argument in [2], $Iz = z$, $Fz = \{z\}$, and $Gz' = \{z\}$, $Jz' = z$. Since G and J are δ -compatible, it follows that $GJz' = JGz'$, and hence that $Gz = Jz$, which leads to z being a common fixed point of the four maps.

Suppose now that F is continuous. Then, as in [2], $\lim Fy_{2n} = Fz$, and $Fz = \{z\}$. Since $F(X) \subseteq J(X)$, there exists a point z' in X such that $Jz' = z$. Applying (1) to $\delta(Gz', Fx_{2n})$ and then taking the limit as $n \rightarrow \infty$ it follows that $Gz' = \{z\}$. Since J and G are δ -compatible, $GJz' = JGz'$, which leads to $Gz = Jz$. Applying (1) to $\delta(Fx_{2n}, Gz)$ and letting $n \rightarrow \infty$, we obtain that $Gz = \{z\}$. Thus $Jz = Gz = \{z\}$.

Since $G(X) \subseteq I(X)$, there exists a point z'' in X such that $Iz'' = z$. Thus

$$\begin{aligned}\delta(Fz'', z) &= \delta(Fz'', Gz) \leq \phi(\delta(Fz'', z), 0, 0, \delta(Fz'', z), 0) \\ &\leq \psi(\delta(Fz'', z), 0, 0, \delta(Fz'', z), 0) \leq \delta(Fz'', z),\end{aligned}$$

which implies that $Fz'' = \{z\}$. Using the fact that F and I are δ -compatible it follows that $Fz = Iz$. Therefore z is a common fixed point of the four maps.

The proofs for J or G continuous are similar and will be omitted.

The uniqueness of z follows from (1). $\square$

Theorems 3.1–3.3 of [2] are special cases of the theorem of this paper.

We remark that Theorem 1 of [1] can also be extended to compatible maps.

References

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