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Special radicals in rings with involution

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Abstract. Special radicals were defined for rings with involution by SALAVOVÀ. In this paper we show that every special radical $\mathcal{R}$ in the variety of rings induces a corresponding special radical $\mathcal{R}_*$ in the variety of rings with involution, and $\mathcal{R}_*(R) \subseteq \mathcal{R}(R)$ for any involution ring R. The reverse inclusion does not hold in general. This theory gives new characterisations for certain concrete radicals.

1. Preliminaries

We recall that an *involution* on a ring R is a mapping $x \longrightarrow x^*(x \in R)$ such that $(x+y)^* = x^*+y^*, (xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. If $x \in R$ and $x^* = x$, then x is called a *symmetric* element of R. A $*$-*ideal* of R is an ideal of the ring R which is closed with respect to the $*$ operation. If R, S are rings with involution, and $f : R \longrightarrow S$ is a ring *homomorphism* (*isomorphism*) such that $f(x^*) = (f(x))^*$ for all $x \in R$, then f is called a $*$-*homomorphism* ($*$-*isomorphism*). Factor involution rings are defined as for rings, and the usual isomorphism theorems may be proved. If R is any ring and R^{op} is its antiisomorphic image, then $R \oplus R^{\mathrm{op}}$ is a ring with involution with $(x, y)^* := (y, x)$ for all $x, y \in R$. This involution is called the *exchange involution*. For further properties of rings with involution, we refer to [4].

If R is a ring, the notation $A \triangleleft R$ means "A is an ideal of R". If $E \triangleleft R$ and $0 \neq A \triangleleft R$ implies $A \cap E \neq 0$, then E is an *essential* ideal of R, denoted $E \triangleleft \cdot R$. Similarly, if R is a ring with involution, $A \triangleleft *R$ means "A is a $*$-ideal of R". If $E \triangleleft *R$ and $0 \neq A \triangleleft *R$ implies $A \cap E \neq 0$, then E is an *essential* $*$-ideal of R, denoted $E \triangleleft * \cdot R$. In the sequel, the

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varieties of rings and rings with involution will be denoted $\underline{\text{Rng}}$ and $\underline{\text{IR}}$, respectively. For a detailed treatment of radical theory in $\underline{\text{Rng}}$ (and related varieties, e.g. $\underline{\text{IR}}$), we refer for example to [12]. We recall that a class $\mathcal{M}$ of prime rings is called *special* if $\mathcal{M}$ is *hereditary* (i.e. $A \lhd R \in \mathcal{M}$ implies $A \in \mathcal{M}$) and *essentially closed* (i.e. $E \lhd \cdot R$, $E \in \mathcal{M}$ implies $R \in \mathcal{M}$). This definition is due to Heyman and Roos [5]. The upper radical determined by $\mathcal{M}$, $\mathcal{UM} := \{R \mid R$ has no nonzero homomorphic image in $\mathcal{M}\}$ is called a *special radical*. In this case

$$\mathcal{UM}(R) = \bigcap\{P \lhd R \mid R/P \in \mathcal{M}\}.$$

Salavovà [8] introduced special radicals for rings with involution. If $R \in \underline{\text{IR}}$, then R is $*$-*prime* if $A, B \lhd *R$, $AB = 0$ implies $A = 0$ or $B = 0$. A class $\mathcal{M}$ in $\underline{\text{IR}}$ is *special* if

S1 : $\mathcal{M}$ consists of $*$-prime involution rings.
S2 : $\mathcal{M}$ is $*$-*hereditary*, i.e. $A \lhd *R \in \mathcal{M}$ implies $A \in \mathcal{M}$.
S3 : $A \lhd *R$, $A \in \mathcal{M}$, R $*$-prime implies $R \in \mathcal{M}$.

Using proofs similar to those of [5] for the ring case, it may be shown that condition S3 may be replaced by

S4 : $\mathcal{M}$ is $*$-*essentialy closed*, i.e. $E \lhd * \cdot R$, $E \in \mathcal{M}$ implies $R \in \mathcal{M}$.

If $\mathcal{M}$ is a special class in $\underline{\text{IR}}$, the upper radical determined by $\mathcal{M}$, $\mathcal{U}_*\mathcal{M} := \{R \mid R$ has no nonzero $*$-homomorphic image in $\mathcal{M}\}$, is called a *special radical*. In this case, as for rings it is easily shown that

$$\mathcal{U}_*\mathcal{M}(R) = \bigcap\{P \lhd *R \mid R/P \in \mathcal{M}\}$$

for all $R \in \underline{\text{IR}}$.

Lee and Wiegandt [6] have noted that not every special radical class $\mathcal{R}$ in $\underline{\text{Rng}}$ has the property that $\mathcal{R}(R) \lhd *R$ for all $R \in \underline{\text{IR}}$, although most of the well-known specials do have this property. Examples of special radicals which do not, are the *right strongly prime* [3], and *superprime* [11] radicals. The following result gives necessary and sufficient conditions for a special radical in $\underline{\text{Rng}}$ to have the above-mentioned property, and hence to be usable as a radical in $\underline{\text{IR}}$.

Proposition 1.1 (cf. [6]). *Let $R = \mathcal{UM}$, where $\mathcal{M}$ is a special class in $\underline{\text{Rng}}$. Then the following are equivalent:*

(a) $\mathcal{R}(R) \lhd *R \quad \forall R \in \underline{\text{IR}}$
(b) $\mathcal{R}(R)^* = \mathcal{R}(R) \quad \forall R \in \underline{\text{IR}}$

(c) $R \in \mathcal{R} \Longrightarrow R^{\mathrm{op}} \in \mathcal{R} \quad \forall R \in \underline{\mathrm{Rng}}$

(d) $R \in \mathcal{M} \Longrightarrow \mathcal{R}(R^{\mathrm{op}}) = 0 \quad \forall R \in \underline{\mathrm{Rng}}$.

Proof. (a) $\Leftrightarrow$ (b) is obvious. (b) $\Leftrightarrow$ (c) is [6], Theorem 1. (c) $\Longrightarrow$ (d) follows from [6], Corollary 1. Hence we need only show (d) $\Longrightarrow$ (c). Let $R \in \mathcal{R}$ and suppose $R^{\mathrm{op}} \notin \mathcal{R}$. Then R^{op} has a nonzero homomorphic image, R^{op}/A, say, which is in $\mathcal{M}$. Hence $\mathcal{R}(R^{\mathrm{op}}/A)^{\mathrm{op}} = 0$, i.e. $\mathcal{R}(R/A) = 0$. This is impossible, since $R \in \mathcal{R}$ and $\mathcal{R}$ is homomorphically closed. Hence $R^{\mathrm{op}} \in \mathcal{R}$, and (c) holds.

Special radicals satisfying the conditions of Proposition 1.1 will be called *symmetric*.

2. Special radicals in rings with involution

In this section we wil show that every special radical in $\underline{\mathrm{Rng}}$ induces a uniquely determined special radical in $\underline{\mathrm{IR}}$.

Lemma 2.1 ([7], Proposition 2.13.35). *Let $R \in \underline{\mathrm{IR}}$. Then R is $*$-prime if and only if there exists a prime ideal P of the ring R such that $P \cap P^* = 0$. Moreover, P may be chosen to be maximal in the class of ideals I of R such that $I \cap I^* = 0$.*

Let $\mathcal{M}$ a special class in $\underline{\mathrm{Rng}}$. Then we define

$$\mathcal{M}^* = \{R \in \underline{\mathrm{IR}} \mid \exists P \triangleleft R \text{ with } P \cap P^* = 0 \text{ and } R/P \in \mathcal{M}\}.$$

Theoem 2.2. *Let $\mathcal{M}$ be a special class in $\underline{\mathrm{Rng}}$. Then $\mathcal{M}^*$ is a special class in $\underline{\mathrm{IR}}$.*

Proof. Let $R \in \mathcal{M}^*$, and let $P \triangleleft R$ be such that $P \cap P^* = 0$ and $R/P \in \mathcal{M}$. Then R/P is prime, whence R is $*$-prime by Lemma 2.1. Hence $\mathcal{M}^*$ satisfies S1. Now suppose that $A \triangleleft * R$. Then $A \cap P$ is a prime ideal of A, and $(A \cap P) \cap (A \cap P)^* = (A \cap P) \cap (A \cap P^*) = A \cap P \cap P^* = 0$. Moreover $A/(A \cap P) \cong (A + P)/P \triangleleft R/P \in \mathcal{M}$, whence $A/(A \cap P) \in \mathcal{M}$, since $\mathcal{M}$ is hereditary. Thus $A \in \mathcal{M}^*$. Let E be an essential $*$-ideal of a ring with involution R. Let $P \triangleleft E$ with $E/P \in \mathcal{M}$, $P \cap P^* = 0$. Since $P \triangleleft R \triangleleft R$ and E/P is prime, $P \triangleleft R$. We will show that $E/P \triangleleft \cdot R/P$. If $P = E$, $P^* = E^* = E$ so $E = P \cap P^* = 0$. Since E is an essential $*$-ideal of R, $R = 0$ and so $E/P \triangleleft \cdot R/P$ trivially. Suppose that $P \subset E$. Now let $0 \neq U \triangleleft R/P$. Then $U = I/P$ for some ideal I of R which properly contains P. Suppose $E \cap I \subseteq P$. Then $EI \subseteq E \cap I \subseteq P$, whence $E \subseteq P$ or $I \subseteq P$ since P is prime. This is impossible, since $P \subset I$ and $P \subset E$. Thus

$E \cap I \not\subseteq P$. Let $x \in (E \cap I) - P$. Then $x + P \in (E/P) \cap (I/P)$. Hence $E/P \triangleleft \cdot R/P$. Since $\mathcal{M}$ is essentially closed, $R/P \in \mathcal{M}$, whence $R \in \mathcal{M}^*$. Thus M^* satisfies S4.

Theorem 2.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be special classes in $\underline{\text{Rng}}$. If $\mathcal{U}\mathcal{M}_1 = \mathcal{U}\mathcal{M}_2$, then $\mathcal{U}_*\mathcal{M}_1^* = \mathcal{U}_*\mathcal{M}_2^*$ in $\underline{\text{IR}}$.*

Proof. Suppose that R is a ring with involution and $R \notin \mathcal{U}_*\mathcal{M}_1^*$. Then R has a nonzero $*$-homomorphic image R' say, in $\mathcal{U}_*\mathcal{M}_1^*$. Then there exists $P \triangleleft R'$ such that $R/P \in \mathcal{M}_1$ and $P \cap P^* = 0$. If $P = R'$ then $P^* = (R')^* = R'$, whence $R' = P \cap P^* = 0$. This is impossible by our choice of R', so $P \neq R'$. Thus R'/P is a nonzero homomorphic image of R' in $\mathcal{M}_1$. Hence $R' \notin \mathcal{U}\mathcal{M}_1$. Since $\mathcal{U}\mathcal{M}_1 = \mathcal{U}\mathcal{M}_2$, R' has a nonzero homomorphic image S, say, in M_2. Clearly, S is a homomorphic image of R, whence $S \cong R/Q$ for some proper ideal Q of the ring R. Consider the involution ring $R/(Q \cap Q^*)$. Then $Q/(Q \cap Q^*) \triangleleft R/(Q \cap Q^*)$ and $(Q/(Q \cap Q^*)) \cap (Q/(Q \cap Q^*))^* = 0$.

Moreover $(R/(Q \cap Q^*))/(Q/(Q \cap Q^*)) \cong R/Q \in \mathcal{M}_2$. Hence $(R/(Q \cap Q^*)) \in \mathcal{M}_2^*$, so $R \notin \mathcal{U}_*\mathcal{M}_2^*$. Thus $\mathcal{U}_*\mathcal{M}_2^* \subseteq \mathcal{U}_*\mathcal{M}_1^*$. The reverse inclusion is proved similarly.

Theorems 2.2 and 2.3 show that every special radical $\mathcal{R}$ in $\underline{\text{Rng}}$ induces a uniquely determined special radical in $\underline{\text{IR}}$. If $\mathcal{M}$ is a special class in $\underline{\text{Rng}}$ and $\mathcal{R} = \mathcal{U}\mathcal{M}$, then we shall denote $\mathcal{U}_*\mathcal{M}^*$ by R_* .

Lemma 2.4. *Let $R \in \underline{\text{IR}}$. A subset Q of R is a $*$-ideal of R such that $R/Q \in \mathcal{M}^*$ if and only if $Q = P \cap P^*$ for some ideal P of the ring R such that $R/P \in \mathcal{M}$.*

Proof. Suppose $Q = P \cap P^*$, where $P \triangleleft R$, $R/P \in \mathcal{M}$. Then $(R/Q)/(P/Q) \cong R/P \in \mathcal{M}$ and $(P/Q) \cap (P/Q)^* = (P \cap P^*)/Q = 0$. Hence $R/Q \in \mathcal{M}^*$. Conversely, suppose that $Q \triangleleft * R$ and that $R/Q \in \mathcal{M}^*$. Then there exists $U \triangleleft R/Q$ such that $(R/Q)/U \in \mathcal{M}$ and $U \cap U^* = 0$. Then $U = P/Q$ for some ideal P of the ring R such that $Q \subseteq P$. Then $R/P \cong (R/Q)/(P/Q) \in \mathcal{M}$ and since $U \cap U^* = 0, P \cap P^* = Q$.

Proposition 2.5. *If $\mathcal{R}$ is a special radical in* $\underline{\text{Rng}}$, *then* $\mathcal{R}_*(R) = \mathcal{R}(R) \cap (\mathcal{R}(R))^*$ *for any* $R \in \underline{\text{IR}}$.

Proof. Let $\mathcal{R} = \mathcal{U}\mathcal{M}$, where $\mathcal{M}$ is a special class in $\underline{\text{Rng}}$. Then

$$\begin{aligned}
\mathcal{R}_*(R) &= \bigcap\{Q \triangleleft *R : R/Q \in \mathcal{M}^*\} \\
&= \bigcap\{P \cap P^* \mid P \triangleleft R \text{ and } R/P \in \mathcal{M}\} \quad \text{(by Lemma 2.4)} \\
&= \bigcap\{P \triangleleft R : R/P \in \mathcal{M}\} \cap \{P^* \mid P \triangleleft R \text{ and } R/P \in \mathcal{M}\} \\
&= \mathcal{R}(R) \cap (\mathcal{R}(R))^*.
\end{aligned}$$

Corollary 2.6. *If $\mathcal{R}$ is a symmetric special radical in* $\underline{\text{Rng}}$, *then* $\mathcal{R}_*(R) = \mathcal{R}(R)$ *for all* $R \in \underline{\text{IR}}$.

3. Examples

From [6] we have that if $\mathcal{R}$ denotes the prime or Jacobson radical for rings then $\mathcal{R}(R)^* = \mathcal{R}(R)$ for any ring with involution R. Moreover, the definitions of $*$-prime and $*$-primitive involution rings coincide with those obtained from the corresponding classes in $\underline{\text{Rng}}$. We refer to [7] for details.

The nil radical

Van der Walt [10] characterised the nil radical $\mathcal{N}$ in $\underline{\text{Rng}}$ as the upper radical determined by the (special) class of *s-prime* rings. A ring R is s-prime if there exists a multiplicatively closed subset S of $R - \{0\}$ such that $0 \neq A \triangleleft R$ implies $A \cap S \neq \emptyset$. If $P \triangleleft R$, then P is called an s-prime ideal of R if R/P is an s-prime ring. The s-prime ideals of R may easily be characterised as follows:

Lemma 3.1. *Let R be a ring and $P \triangleleft R$. Then the following are equivalent:*

(a) *P is an s-prime ideal of R.*

(b) *$R - P$ contains a multiplicatively closed subset S such that $A \triangleleft R$, $A \not\subseteq P$ implies $A \cap S \neq \emptyset$.*

(c) *$R - P$ contains a multiplicatively closed subset S such that $A \triangleleft R$, $P \subset A$ implies $A \cap S \neq \emptyset$.*

Let $\mathcal{M}$ be the class of s-prime rings. Clearly $R \in \mathcal{M}$ implies that $R^{\mathrm{op}} \in \mathcal{M}$. Hence $\mathcal{N}$ is a symmetric special radical by Proposition 1.1.

If $R \in \underline{\text{IR}}$, then R is called $*$-s-prime if there exists a multiplicatively closed subset S of R such that $0 \neq I \triangleleft *R$ implies $I \cap S \neq \emptyset$.

Lemma 3.2. *Let $R \in \underline{\mathrm{IR}}$. Then R is $*$-s-prime if and only if there exists an s-prime ideal P of R such that $P \cap P^* = 0$.*

Proof. Let R be $*$-s-prime and let S be the required multiplicatively closed system in $R-\{0\}$. If $0 \neq I,\ J \triangleleft * R$, there exists $s \in I \cap S$, $s' \in J \cap S$, and $ss' \in S$ whence $ss' \neq 0$. Thus $IJ \neq 0$. Hence, R is $*$-prime. By Lemma 2.1, there exists a prime ideal P of $\mathcal{R}$ such that $P \cap P^* = 0$ and P is maximal with respect to this property. We show that either $S \cap P = \emptyset$ or $S^* \cap P = \emptyset$. For suppose $s \in S \cap P$, $s' \in S \cap P^*$. Then $ss' \in S$ and $ss' \in PP^* \subseteq P \cap P^* = 0$. Thus $0 \in S$, which is clearly impossible. Hence $S \cap P = \emptyset$, or $S \cap P^* = \emptyset$, so $S \cap P = \emptyset$ or $S^* \cap P = \emptyset$. Since both S and S^* are multiplicatively closed, we may assume $S \cap P \neq \emptyset$. Let $P \subset I \triangleleft R$. By maximality of P, $I \cap I^* \neq 0$. Since $I \cap I^* \triangleleft * R$, $I \cap I^* \cap S \neq \emptyset$, whence $I \cap S \neq \emptyset$. Thus P is an s-prime ideal of R.

Conversely, let $R \in \underline{\mathrm{IR}}$ and let P be an s-prime ideal of the ring R such that $P \cap P^* = 0$. Then by Lemma 3.1 (b), there exists a multiplicatively closed subset S of $R - P$ such that $I \triangleleft R$, $I \nsubseteq P$ implies $I \cap S \neq \emptyset$. Let $0 \neq A \triangleleft * R$. If $A \subseteq P$, then $A^* \subseteq P^*$, i.e. $A \subseteq P^*$ whence $A \subseteq P \cap P^* = 0$. Thus $A \nsubseteq P$, whence $A \cap S \neq \emptyset$. Hence R is $*$-s-prime.

The above Lemma shows that the class of $*$-s-prime involution rings is precisely the class obtained by applying the techniques of Section 2 to the class of s-prime rings. Hence this class is special in $\underline{\mathrm{IR}}$. Since $\mathcal{N}$ is symmetetric, we have:

Proposition 3.3. $\mathcal{N}(R) = \bigcap\{P \triangleleft R \mid R/P$ *is* $*$-s-*prime*$\}$ *for all* $R \in \underline{\mathrm{IR}}$.

We note that $\underline{\mathrm{IR}}$ is a variety of Ω-groups [2] where Ω consists of the multiplication and $*$-operators. Buys and Gerber [1] introduced a concept of nilpotence for Ω-groups. An element x of $R \in \underline{\mathrm{IR}}$ is $*$-nilpotent in the sense if there exists $n \in \mathbb{N}$ such that $x_1 \dots x_n = 0$, where $x_i = x$ or x^* for $1 \leq i \leq n$. If $I \triangleleft * R$, we define I to be *weakly nil* if every element of I is $*$-nilpotent, and

$$\mathcal{N}_W(R) = \sum\{I \triangleleft R \mid I \text{ is weakly nil}\}.$$

Clearly, $\mathcal{N}(R) \subseteq \mathcal{N}_W(R)$.

Lemma 3.4. *Let $I \triangleleft *R \in \underline{\mathrm{IR}}$. Then the following are equivalent:*

(a) *I is weakly nil.*

(b) *Every symmetric element of I is nilpotent.*

(c) *xx^* is nilpotent for all $x \in I$.*

PROOF. (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (c) are trivial. Suppose (c) holds. Let $x \in I$. Then $(xx^*)^n = 0$ for some $n \in \mathbb{N}$, i.e. $xx^*xx^* \dots xx^* = 0$. Hence I is weakly nil.

Remark. It follows from Lemma 3.4 that the inclusion $\mathcal{N}_W(R) \subseteq \mathcal{N}(R)$ is equivalent to a conjecture of McCrimmon [4]: If every symmetric element of $R \in \underline{\mathrm{IR}}$ is nilpotent, then R is a nil ring.

A ring with involution R is called *strongly s-prime* if there exists a subset S of $R - \{0\}$ which is closed with respect to the multiplication and $*$-operators such that $0 \neq I \triangleleft *R$ implies $I \cap S \neq \emptyset$. It follows from [2], Theorem 3.29 that the class $\mathcal{M}_S$ of strongly s-prime involution rings is special. From [1], Theorem 3.13 and Corollary 3.16, we have

Proposition 3.5. *Let $R \in \underline{\mathrm{IR}}$. Then*

$$\mathcal{N}_W(R) = \bigcap\{I \triangleleft *R \mid R/I \text{ is strongoly } s\text{-prime}\}.$$

We do not know whether or not $\mathcal{N}_W$ can be obtained from a special radical class of rings using the methods of Section 2. However, we do have the following result.

Proposition 3.6. *If there exists a hereditary symmetric radical class of rings $\mathcal{R}$ such that $\mathcal{N}_W(R) = \mathcal{R}(R) \cap \mathcal{R}(R)^*$ for every $R \in \underline{\mathrm{IR}}$, then $\mathcal{R} = \mathcal{N}$.*

PROOF. Let $R \in \mathcal{N}$. Then $R^{\mathrm{op}} \in \mathcal{N}$, whence $R \oplus R^{\mathrm{op}} \in \mathcal{N}_W$. Hence $\mathcal{R}(R \oplus R^{\mathrm{op}}) = R \oplus R^{\mathrm{op}}$, so $R \oplus R^{\mathrm{op}} \in \mathcal{R}$. Since $R \cong (R, 0) \triangleleft R \oplus R^{\mathrm{op}}$ and $\mathcal{R}$ is hereditary, $R \in \mathcal{R}$. Hence $\mathcal{N} \subseteq \mathcal{R}$. Conversely, let $R \in \mathcal{R}$. Since $\mathcal{R}$ is symmetric, $R^{\mathrm{op}} \in \mathcal{R}$. Hence $R \oplus R^{\mathrm{op}} \in \mathcal{R}$, so $\mathcal{N}_W(R \oplus R^{\mathrm{op}}) = \mathcal{R}(R \oplus R^{\mathrm{op}}) \cap \mathcal{R}(R \oplus R^{\mathrm{op}})^* = R \oplus R^{\mathrm{op}}$. If $x \in R$, (x, x) is a symmetric element of $R \oplus R^{\mathrm{op}}$ whence $(x, x)^n = (0, 0)$ for some $n \in \mathbb{N}$, by Lemma 3.4 (b), and so $x^n = 0$. Thus $R \in \mathcal{N}$ so $\mathcal{R} \subseteq \mathcal{N}$.

The antisimple radical

It is well known that a ring R is *subdirectly irreducible* if and only if the intersection of the nonzero ideals of R is nonzero. The intersection $H(R)$ is called the *heart* of R. Simiarly, if $R \in \underline{\mathrm{IR}}$, then R is subdirectly irreducible in $\underline{\text{IR}}$ if and only if the intersection of the nonzero $*$-ideals of R is nonzero. This intersection is denoted $H_*(R)$.

The *antisimple radical* $\mathcal{A}$ in $\underline{\text{Rng}}$ is the upper radical determined by the special class of all prime subdirectly irreducible (psdi) rings [9]. Accordingly, we will define a $*$-psdi involution ring to be one which is subdirectly irreducible and prime. As for rings these are precisely the involution rings R which are subdirectly irreducible and for which $H_*(R)$ is idempotent.

Lemma 3.7. *Let $R \in \underline{\text{IR}}$. Then R is $*$-psdi if and only if there exists $P \triangleleft R$ such that $P \cap P^* = 0$ and R/P is a psdi ring.*

Proof. Let R be $*$-psdi. Since R is $*$-prime, there exists a prime ideal P of R such that $P \cap P^* = 0$, and P is maximal with respect to this property. Let $H = H_*(R)$. Then $H \nsubseteq P$ for if $H \subseteq P$, then $H = H^* \subseteq P^*$, whence $H \subseteq P \cap P^* = 0$. This is impossible since R is $*$-psdi. Hence $H \nsubseteq P$, so $0 \neq (H + P)/P \triangleleft R/P$. We claim $H(R/P) = (H + P)/P$. For if $0 \neq U \triangleleft R/P$, then $U = I/P$, where $P \subset I \triangleleft R$. By our choice of P, $I \cap I^* \neq 0$. Since $I \cap I^* \triangleleft *R$, $H \subseteq I \cap I^* \subseteq I$, so $H + P \subseteq I$ whence $(H+P)/P \subseteq I/P$. Thus $H(R/P) = (H+P)/P$ and so R/P is subdirectly irreducible. Since R/P is prime, it is psdi, as required.

Conversely, Let $P \triangleleft R$ be such that $P \cap P^* = 0$ and R/P is psdi. Let $H(R/P) = H/P$, where $P \subset H \triangleleft R$. If $H \subseteq P^*$, then $H^* \subseteq P \subset H$, whence $H^* \subset H$, which is impossible. Hence, $H \nsubseteq P^*$, whence $H^* \nsubseteq P$. Since P is prime, $HH^* \nsubseteq P$, whence $HH^* \neq 0$. Suppose $0 \neq I \triangleleft *R$. As before, $I \nsubseteq P$, whence $P \subset I + P$, so $0 \neq (I + P)/P \triangleleft R/P$. Hence, $H/P \subseteq (I + P)/P$ and so $H \subseteq I + P$. Thus $HH^* \subseteq$
$q(I+P)(I^*+P^*) = (I+P)(I+P^*) = I^2+PI+IP^*+PP^* = I^2+PI+IP^* \subseteq I$ (since $PP^* \subseteq P \cap P^* = 0$). It follows that $H_*(R) = HH^*$, and so R is subdirectly irreducible in $\underline{\text{IR}}$. Since P is a prime ideal of R, R/P is $*$-prime by Lemma 2.1. Henece R is $*$-psdi.

As before, the above Lemma shows that the class of $*$-psdi involution rings is special. It is easily seen that if R is psdi, then so is R^{op}, and so $\mathcal{A}$ is symmetric. Hence:

Proposition 3.8. $\mathcal{A}(R) = \bigcap\{P \triangleleft R \mid R/P$ *is $*$-psdi*$\}$ *for all* $R \in \underline{\text{IR}}$.

The Behrens radical

Let $\mathcal{M}$ be the class of all psdi rings whose hearts contain a nonzero idempotent element. Then $\mathcal{M}$ is a special class, and the upper radical $\mathcal{B}$ determined by $\mathcal{M}$ is known as the *Behrens radical*. We refer to [9] for details of this radical. Clearly, $\mathcal{B}$ is symmetric.

Proposition 3.9. *Let $R \in \underline{\mathrm{IR}}$. Then $R \in \mathcal{M}^*$ if and only if R is $*$-psdi and $H_*(R)$ contains a nonzero idempotent element.*

Proof. Suppose $R \in \mathcal{M}^*$. Then there exists $P \lhd R$ such that $P \cap P^* = 0$ and $R/P \in \mathcal{M}$. Since R/P is psdi, R is $*$-psdi by Lemma 3.7. Let $H(R/P) = H/P$, where $P \subset H \lhd R$. By the proof of Lemma 3.7, $H_*(R) = HH^*$. If $P = 0$, then $H = H(R)$, whence $H \subseteq HH^*$. But $HH^* \subseteq H$, so $H(R) = H_*(R)$ has a nonzero idempotent element, so does $H_*(R)$. Suppose $P \neq 0$. Let $e \in H$ be such that $0 \neq e + P = (e+P)^2$. Since $P \neq 0$, $P^* \nsubseteq P$, so $0 \neq (P+P^*)/P \lhd R/P$. Hence, $e + P \in H/P \subseteq (P+P^*)/P$, and so there exists $p \in P$ such that $e + P = p^* + P$. Hence $(p^*+P)^2 = p^* + P$ whence $(p^*)^2 - p^* \in P$. Since $(p^*)^2 - p^* \in P^*$ and $P \cap P^* = 0$, $(p^*)^2 = p^*$. It follows that $p^2 = p$. Let $f = p + p^*$. Then $f^2 = (p+p^*)^2 = p^2 + pp^* + pp^* + (p^*)^2 = p^2 + (p^*)^2$ (since pp^*, $p^*p \in P \cap P^* = 0$) $= p + p^* = f$. Hence f is idempotent. Moreover, $f + P = p + p^* + P = p^* + P = e + P$. Hence $f - e \in P \subseteq H$. Since $e \in H$, $f \in H$. Moreover, $f^* = (p+p^*)^* = p^* + p = p + p^* = f$. Moreover, $f = f^* = ff^* \in HH^* = H_*(R)$. Thus $H_*(R)$ contains an idempotent element.

Conversely, suppose R is $*$-psdi and that $H = H_*(R)$ contains an idempotent element, e, say. Then by Lemma 3.7, there exists $P \lhd R$ such that R/P is psdi and $P \cap P^* = 0$. Moreover, by the proof of that Lemma, $H(R/P) = (H+P)/P$. Then $e + P \in H(R/P)$ and $(e+P)^2 = e^2 + P = e + P$, so $R/P \in \mathcal{M}$. Hence $R \in \mathcal{M}^*$.

Corollary 3.10. $\mathcal{B}(R) = \bigcap\{P \lhd R \mid R/P$ *is $*$-psdi and $H_*(R/P)$ contains a nonzero idempotent element*$\}$ *for all $R \in \underline{\mathrm{IR}}$.*

The Brown–McCoy radical

It is well known that the *Brown–McCoy radical* for rings is the upper radical $\mathcal{G}$ determined by the class of simple rings with unity. An ideal P of a ring R is called modular if the factor ring R/P has a unity. Clearly, R/P is a simple ring with unity if and only if P is a maximal modular ideal of R. An involution ring R will be caled $*$-*simple* if R has no $*$-ideals except $\{0\}$ and R.

Lemma 3.11 ([7] Lemma 2.13.23)**.** *Let $R \in \underline{\mathrm{IR}}$. Let R is $*$-simple if and only if there exists a maximal ideal P of R such that $P \cap P^* = 0$.*

Theorem 3.12. *Let $R \in \underline{\mathrm{IR}}$. Then the following are equivalent:*

(a) *R is $*$-simple with unity.*

(b) *There exists a modular maximal ideal P of the ring R such that $P \cap P^* = 0$.*

(c) *Either R is a simple ring with unity or R is $*$-isomorphic to $S \oplus S^{\mathrm{op}}$ with the exchange involution, where S is a simple ring with unity.*

Proof. (a) $\Longrightarrow$ (b): Since R is $*$-simple, by Lemma 3.11 there exists a maximal ideal P of the ring R such that $P \cap P^* = 0$. Let e be the identity of R. Then $e + P$ is the identity of R/P. Hence P is modular.

(b) $\Longrightarrow$ (c): Let P be a modular maximal ideal of the ring R such that $P \cap P^* = 0$. If $P = 0$, then R is a simple ring with unity. Suppose $P \neq 0$. Then $P \neq P^*$, whence $P \subset P + P^*$. By maximality of P, $R = P + P^*$. It follows that R is isomorphic to $P \oplus P^* \cong P \oplus P^{\mathrm{op}}$. The isomorphism $\alpha : R \to P \oplus P^*$ is given by $\alpha(x) = (p, q)$, where p and q are the unique elements of P and P^* respectively such that $x = p + q$. Then $(\alpha(x))^* = (p, q)^* = (q^*, p^*) = \alpha(x^*)$. Hence α is a $*$-isomorphism of R onto $P \oplus P^*$. Now P is isomorphic to R/P^*, which is a simple ring with unity since P^* is a modular maximal ideal of R. Hence, P is a simple ring with unity, as required.

(c) $\Longrightarrow$ (a): If R is a simple ring with unity, it is $*$-simple. If $R = S \oplus S^{\mathrm{op}}$ where S is a simple ring with unity, then the $*$-ideals of R are of the form $A \oplus A^{\mathrm{op}}$, where $A \triangleleft S$. Since S is simple, R is $*$-simple. Since S has a unity, so does S^{op}. Hence R has a unity.

In view of Theorem 3.11, and the fact that $\mathcal{G}$ is clearly symmetric, we have:

Proposition 4.9. $\mathcal{G}(R) = \bigcap\{P \triangleleft R \mid R/P$ *is $*$-simple with unity*$\}$ *for all $R \in \underline{\mathrm{IR}}$.*

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