

On exceptional values of meromorphic functions

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1. Let $f(z)$ be a meromorphic function. For $0 \leq |\alpha| < \infty$ let $n(r, \alpha)$ denote the number of roots of $f(z) - \alpha$ in $|z| \leq r$. Further let $\bar{n}(r, \alpha)$ denote the number of roots of $f(z) - \alpha = 0$ in $|z| \leq r$ each root being counted only once. For $\alpha = \infty$ let $n(r, \infty) = n(r, f)$ be the number of poles of $f(z)$ in $|z| \leq r$ and let $\bar{n}(r, \infty) = \bar{n}(r, f)$ be the number of poles of $f(z)$ in $|z| \leq r$ each pole being counted only once. Let

$$N\left(r, \frac{1}{f-\alpha}\right) = N(r, \alpha) = \int_0^r \frac{n(t, \alpha) - n(0, \alpha)}{t} dt + n(0, \alpha) \log r$$

and

$$N(r, \infty) = N(r, f) = \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} dt + n(0, \infty) \log r.$$

Let

$$\bar{N}(r, \alpha) = \int_0^r \frac{\bar{n}(t, \alpha) - \bar{n}(0, \alpha)}{t} dt + \bar{n}(0, \alpha) \log r$$

and

$$\bar{N}(r, \infty) = \bar{N}(r, f) = \int_0^r \frac{\bar{n}(t, \infty) - \bar{n}(0, \infty)}{t} dt + \bar{n}(0, \infty) \log r.$$

Then clearly $n(r, f') = n(r, f) + \bar{n}(r, f)$ and $N(r, f') = N(r, f) + \bar{N}(r, f)$ see NEVANLINNA [1, 105].

Let $T(r, f) = T(r)$ be the NEVANLINNA characteristic function. Let E denote the set of positive non-decreasing functions Φ such that

$$\int_A^\infty \frac{dx}{x\Phi(x)} \text{ is convergent.}$$

Then we say that α is an e. v. E if

$$\liminf_{r \rightarrow \infty} \frac{T(r)}{n(r, \alpha)\Phi(r)} > 0 \quad \text{for some } \Phi \in E.$$

See S. M. SHAH [2].

If

$$\delta(r, \alpha) = \delta(\alpha) = 1 - \limsup_{r \rightarrow \infty} \frac{N(r, \alpha)}{T(r, f)} > 0,$$

then α will be called e. v. N (exceptional value in the sense of NEVANLINNA), see [1, p. 79]; and if

$$\Delta(r, \alpha) = \Delta(\alpha) = 1 - \liminf_{r \rightarrow \infty} \frac{N(r, \alpha)}{T(r, f)} > 0$$

then α will be called e. v. V. (in the sense of Valiron). Also let

$$\theta(\alpha) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, \alpha)}{T(r)}$$

and

$$\mu(r, \alpha) = \mu(\alpha) = \liminf_{r \rightarrow \infty} \frac{N(r, \alpha) - \bar{N}(r, \alpha)}{T(r)}$$

2. We prove the following theorem.

Theorem 1. Let $f(z)$ be a meromorphic function such that

$$\sum_1^{\infty} \delta(a_i) + \sum_1^{\infty} \mu(a_i) = 1,$$

where $a_1, a_2, \dots$ are any finite constants pairwise different, then, if " ∞ " is an e. v. E for $f(z)$, it is also an e. v. E for $f'(z)$ and conversely.

PROOF. ∞ is an e. v. E for $f(z)$, so $\delta(\infty) = 1$, see S. M. SHAH [2]. But, it is known that

$$(2.1) \quad \limsup_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \leq 2 - \delta(\infty) - \mu(\infty) \dots = 1,$$

see NEVANLINNA [1, 104]. Also it is known that

$$(2.2) \quad \liminf_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \geq \sum_1^{\infty} \delta(a_i) + \sum_1^{\infty} \mu(a_i) = 1,$$

see WITTICH [6, 21]. From (2.1) and (2.2) it follows that

$$T(r, f') \sim T(r, f).$$

Now

$$n(r, f') = n(r, f) + \bar{n}(r, f) \leq 2n(r, f)$$

Hence

$$\frac{T(r, f')}{n(r, f')\Phi(r)} \geq \frac{1}{2} \frac{T(r, f')}{n(r, f)\Phi(r)} > \frac{1}{2} \frac{(1-\varepsilon)T(r, f)}{n(r, f)\Phi(r)} \quad \text{for } r \geq r_0.$$

Since " ∞ " is an e. v. E for $f(z)$, it follows

$$\liminf_{r \rightarrow \infty} \frac{T(r, f')}{n(r, f')\Phi(r)} > 0.$$

Hence ∞ is an e. v. E for $f'(z)$.

Conversely if ∞ is an e. v. E for $f'(z)$, we prove that it is e. v. E for $f(z)$ also. Since $\delta(f', \infty) = 1$,

$$\lim_{r \rightarrow \infty} \frac{N(r, f')}{T(r, f')} = 0.$$

But $N(r, f') = N(r, f) + \bar{N}(r, f) \geq 2\bar{N}(r, f)$, so

$$\frac{\bar{N}(r, f)}{T(r, f)} \leq \frac{1}{2} \frac{N(r, f')}{T(r, f')} \frac{T(r, f')}{T(r, f)}.$$

$$\limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{T(r, f)} \leq \frac{1}{2} \limsup_{r \rightarrow \infty} \frac{N(r, f')}{T(r, f')} \limsup_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} = 0,$$

so

$$\theta(\infty) = 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}(r, f)}{T(r, f)} = 1.$$

But it is known that [6, 21],

$$\limsup_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \leq 2 - \theta(\infty) = 1.$$

But, as before,

$$\liminf_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \geq \sum_1^{\infty} \delta(r, a_j) + \sum_1^{\infty} \mu(r, a_j) = 1.$$

Hence $T(r, f') \sim T(r, f)$. Now $n(r, f') \geq n(r, f)$. So for $r \geq r_0$,

$$\frac{T(r, f)}{n(r, f)\Phi(r)} > \frac{(1 - \varepsilon)T(r, f')}{n(r, f')\Phi(r)}.$$

Now since ∞ is an e. v. E for $f'(z)$, so the right hand side of the above inequality is positive, hence

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{n(r, f)\Phi(r)} > 0.$$

So ∞ is an e. v. E of $f(z)$.

Theorem 2. If ∞ is an e. v. E for a meromorphic function $f(z)$ and $\sum_{z_i \neq \infty} \delta(z_i) = 1$, then

$$\delta(f^{(k)}, 0) + \delta(f^{(k)}, \infty) = 2 \quad \text{for } k = 1, 2, \dots$$

PROOF. Since ∞ is an e. v. E,

$$\delta(f, \infty) = 1.$$

But

$$\limsup_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \leq 2 - \delta(\infty) - \mu(\infty) = 1.$$

Again

$$\liminf_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \geq \sum_1^{\infty} \delta(z_i) + \sum_1^{\infty} \mu(z_i) = 1.$$

So $T(r, f') \sim T(r, f)$.

As in Theorem 1, since ∞ is an e. v. E for $f'(z)$, $\delta(f', \infty) = 1$. Again, we have [6, 18],

$$N\left(r, \frac{1}{f'}\right) + \sum_1^q m(r, a_j) + S(r) \leq T(r, f').$$

So

$$\begin{aligned} N\left(r, \frac{1}{f'}\right) + m\left(r, \frac{1}{f'}\right) - m\left(r, \frac{1}{f'}\right) + \sum_1^q m(r, a_j) + S(r) &\leq T(r, f'), \\ \sum_1^q m(r, a_j) + S(r) &\leq m\left(r, \frac{1}{f'}\right) + O(1). \end{aligned}$$

Hence

$$\liminf \sum_1^q \frac{m(r, a_j)}{T(r, f')} + \frac{S(r)}{T(r, f')} \leq \delta(f', 0).$$

So

$$\frac{1}{\limsup_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)}} \sum_1^q \delta(r, a_j) \leq \delta(f', 0).$$

But

$$\limsup_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \leq 1,$$

and

$$\sum_1^{\infty} \delta(r, a_j) = 1.$$

Hence

$$\delta(f', 0) = 1,$$

so

$$\delta(f', 0) + \delta(f', \infty) = 2.$$

Repeating the argument we get

$$\delta(f^{(k)}, 0) + \delta(f^{(k)}, \infty) = 2.$$

Corollary. If ∞ is an e. v. E for a meromorphic function $f(z)$ and if further $\sum_1^{\infty} \delta(\alpha_i) = 1$, then q , the order of the function, must be a positive integer.

For an alternative proof of the corollary see S. M. SHAH and S. K. SINGH [5].

PROOF. By theorem 2,

$$\delta(f', 0) = 1, \quad \delta(f', \infty) = 1.$$

Hence

$$\lim_{r \rightarrow \infty} \frac{N(r, 1/f') + N(r, f')}{T(r, f')} = 0.$$

Hence the order of $f'(z)$ must be a positive integer, because for non-integral order,

$$\limsup_{r \rightarrow \infty} \frac{N(r, a) + N(r, b)}{T(r)} > 0 \text{ for all } a \text{ and } b, \quad (a \neq b),$$

see NEVANLINNA [1, 51]. But the order of a meromorphic function is the same as the order of its derivative. This proves the corollary.

Theorem 3. *If there is at least one e. v. N for a meromorphic function $f(z)$, then 0 must be an exceptional value of $f'(z)$ in the sense of VALIRON.*

PROOF. Suppose, if possible 0 is a normal value for $f'(z)$ in the sense of Valiron. Then $\Delta(f', 0) = 0$. But, it is known that

$$\Delta(f', 0) \liminf_{r \rightarrow \infty} \frac{T(r, f')}{T(r, f)} \cong \sum_1^{\infty} \delta(a_i),$$

see [4]. So, $\sum_1^{\infty} \delta(a_i) = 0$, contradicting the hypothesis that $\delta(a_i) > 0$ for at least one a_i . Hence 0 is an exceptional value of $f'(z)$ in the sense of VALIRON.

References

- [1] R. NEVANLINNA, Le théorème de Picard Borel et la théorie des fonctions méromorphes, *Paris*, 1929.
- [2] S. M. SHAH, On exceptional values of entire functions. *Compositio Math.* **9** (1951), 227–238.
- [3] S. M. SHAH, Exceptional values of entire and meromorphic functions, *Duke Math. J.* **19** (1952), 585–594.
- [4] S. M. SHAH and S. K. SINGH, On the derivatives of a meromorphic function with maximum defect, *Math. Z.* **65** (1956), 171–174.
- [5] S. M. SHAH and S. K. SINGH, Meromorphic functions with maximum defect sum, *Tôhoku Math. J.* **11** (1959), 447–452.
- [6] H. WITTICH, Neuere Untersuchungen über eindeutige analytische Funktionen, *Berlin*, 1955.

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