

On Ogawa's Criterion For Univalence*

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1. Introduction

In a recent note, OGAWA obtained the following result ([2], p. 8).

Theorem 1. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| \leq 1$, with $f'(z) \neq 0$ there, let $q(t)$ denote a single-valued real function, sufficiently smooth for all t , and let k denote a real constant. If the inequality

$$(1) \quad \int_{\theta_1}^{\theta_2} d[\arg[e^{i\theta} f'(e^{i\theta})] + k q(f(e^{i\theta}))] > -\pi, \quad z = re^{i\theta},$$

holds for all $\theta_1 < \theta_2$, then $f(z)$ is univalent for $|z| < 1$.

OGAWA then applied Theorem 1 to the special cases (i) $q(f(z)) = \Im(f(z))$ and (ii) $q(f(z)) = \arg f(z)$ to obtain various sufficient conditions for the univalence (or p -valence) of analytic functions $f(z)$ [3].

In this note we obtain certain representations of functions $f(z)$ satisfying an inequality of the form (1) for the two elementary cases alluded to above. These representations yield some results which are modest extensions of OGAWA's original results; they also raise other questions, answers to which should prove interesting.

2. Principal Results

Most of the results of this note depend upon the following lemma which is a simple extension of one due to KAPLAN ([1], p. 174).

Lemma. Let $p(\theta)$ denote a real-valued function that is of bounded variation for $0 \leq \theta \leq 2\pi$, and let $p(\theta)$ satisfy

$$(2) \quad \int_0^{2\pi} dp(\theta) = 2\pi(k+1),$$

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where k is a real constant, and

$$(3) \quad \int_{\theta_1}^{\theta_2} dp(\theta) > -\pi,$$

for all $\theta_1 < \theta_2$. Then there exists a real-valued non-decreasing function $q(\theta)$ such that

$$(4) \quad \int_0^{2\pi} dq(\theta) = 2\pi(k+1)$$

and

$$(5) \quad |p(\theta) - q(\theta)| \leq \frac{\pi}{2}$$

hold. Moreover, if $P(r, \theta)$ and $Q(r, \theta)$ denote the functions

$$(6) \quad \begin{aligned} P(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \Re \left(\frac{1 + ze^{-iz}}{1 - ze^{-iz}} \right) [p(z) - (k+1)z] dz, \quad z = re^{i\theta}, \\ Q(r, \theta) &= \frac{1}{2\pi} \int_0^{2\pi} \Re \left(\frac{1 + ze^{-iz}}{1 - ze^{-iz}} \right) [q(z) - (k+1)z] dz, \quad z = re^{i\theta}, \end{aligned}$$

harmonic for $r < 1$, and if $F(z)$ and $G(z)$ are analytic (not necessarily single-valued) functions satisfying

$$(7) \quad \begin{aligned} P(r, \theta) &= \arg F(z), \quad |F(0)| = 1, \quad \arg 1 = 0, \\ Q(r, \theta) &= \arg G(z), \quad |G(0)| = 1, \quad \arg 1 = 0, \end{aligned}$$

then

$$(8) \quad \Re \left(\frac{F(z)}{G(z)} \right) \geq 0$$

and

$$(9) \quad \frac{\partial}{\partial \theta} \arg [z^{k+1} G(z)] \geq 0$$

hold for $0 < |z| \leq 1$.

PROOF. A proof of this lemma is contained, for all practical purposes, in KAPLAN's note ([1], p. 174), hence we shall omit the details. However, we do note that the condition (9) is the usual one that guarantees that $w = z^{k+1}G(z)$ maps each circle $|z| = r$ onto a curve that is (spiral and) star-like with respect to $w = 0$; if k is a nonnegative integer, then the image of each circle $|z| = r$ under $w = z^{k+1}G(z)$ is a closed (and star-like) curve.

We now apply our lemma.

Theorem 2. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| \leq 1$ with $f'(z) \neq 0$ and $f(z)/z \neq 0$ there, and let m denote a real constant. Then $f(z)$ satisfies the inequality

$$(10) \quad \int_{\theta_1}^{\theta_2} \Re \left[1 + e^{i\theta} \frac{f''(e^{i\theta})}{f'(e^{i\theta})} + me^{i\theta} f'(e^{i\theta}) \right] d\theta > -\pi$$

for all $\theta_1 < \theta_2$ if and only if there exists a close-to-convex function $H(z) = z + \sum_{n=2}^{\infty} b_n z^n$, univalent for $|z| < 1$, such that

$$(11) \quad H(z) = \frac{e^{mf(z)} - 1}{m}, \quad mH(z) + 1 \neq 0.$$

PROOF. We first suppose that there exists some real m such that $f(z)$ satisfies (10) for all $\theta_1 < \theta_2$. We apply the Lemma, with $p(\theta) = \arg [e^{i\theta} f''(e^{i\theta}) e^{mf(e^{i\theta})}]$, with $k=0$, to obtain the functions $F(z)$ and $G(z)$ satisfying (8) and (9). But here it is clear that $F(z) = f'(z) e^{mf(z)}$, so that (9) becomes

$$(12) \quad \Re \left(\frac{f'(z) e^{mf(z)}}{G(z)} \right) \geq 0.$$

Since $G(z)$ satisfies (9) with $k=0$, we conclude that $zG(z)$ is a starlike and univalent function for $|z| < 1$. Hence there exists a convex and univalent function $\Phi(z)$, with $\Phi(0)=0$, $|\Phi'(0)|=1$ such that

$$\Re \left(\frac{f'(z) e^{mf(z)}}{\Phi'(z)} \right) \geq 0$$

holds for $|z| \leq 1$; this implies that $e^{mf(z)}$ is univalent and close-to-convex for $|z| < 1$ ([1], p. 169). One part of the theorem now follows.

The remaining part of the theorem is now a matter of verifying the inequality for a function of the form (11).

Remark 1. If for some real m , $f(z)$ satisfies (10), for all $\theta_1 < \theta_2$, it follows that $H(z)$ in (11) is univalent. Therefore $f(z)$, is univalent too. This is one of OGAWA's results ([2], p. 9).

Remark 2. If $f(z)$ satisfies (10), then the m cannot be arbitrary. For, since $H(z)$ is a univalent function that does not take on the value $-\frac{1}{m}$, it follows from the $\frac{1}{4}$ theorem that $|m| \leq 4$. Indeed, if $H(z)$ is univalent, and if m is chosen so that $mH(z) + 1 \neq 0$, for $|z| < 1$, then $f(z)$ defined by (11) is univalent too; one uses the close-to-convex property of $H(z)$ in order to establish (10).

Theorem 3. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| \leq 1$ with $f'(z) \neq 0$ and $f(z)/z \neq 0$ there, and let m denote a real constant. Then

$$(13) \quad \Re \left[1 + z \frac{f''(z)}{f'(z)} + mzf'(z) \right] \geq 0$$

holds for $|z| \leq 1$ if and only if $f(z)$ is univalent for $|z| < 1$ and has the representation

$$(14) \quad f(z) = \frac{1}{m} \log(1 + m\Phi(z)), \quad 1 + m\Phi(z) \neq 0,$$

where $\Phi(z) = z + \sum_{n=2}^{\infty} c_n z^n$ is a (normalized) univalent convex function.

PROOF. Suppose that (13) holds for $|z| \leq 1$. Then the standard procedure relative to positive harmonic functions yields the HERGLOTZ representation

$$(15) \quad 1 + z \frac{f''(z)}{f'(z)} + mzf'(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-iz}}{1 - ze^{-iz}} d\mu(x)$$

where $\mu(x)$ is non-decreasing and satisfies

$$(16) \quad \int_0^{2\pi} d\mu = 2\pi.$$

From (15) and (16) we obtain, after several simple operations,

$$(17) \quad e^{mf(z)} - 1 = m\Phi(z),$$

where

$$(18) \quad \Phi(z) = \int_0^z e^{-\frac{1}{\pi} \int_0^{2\pi} \log(1 - ze^{-iz}) d\mu(x)} dz.$$

It is well-known that $\Phi(z)$ in (18) is a (normalized) convex function univalent for $|z| < 1$. It now follows that $f(z)$ is also univalent.

The converse can be readily verified by several simple computations.

Remark 3. OGAWA showed that (13) implies $f(z)$ is not only univalent, but also convex in the direction of the imaginary axis ([2], p. 9). This last property of $f(z)$ follows easily from the representation (17); all one must do is study the variation of $\arg f(z)$.

Theorem 4. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic for $|z| \leq 1$, with $f'(z) \neq 0$ and $f(z)/z \neq 0$ there, and let m denote a real constant. Then a necessary and sufficient condition that

$$(19) \quad \int_{\theta_1}^{\theta_2} \Re \left[1 + e^{i\theta} \frac{f''(e^{i\theta})}{f'(e^{i\theta})} + me^{i\theta} \frac{f''(e^{i\theta})}{f(e^{i\theta})} \right] d\theta > -\pi$$

hold for all $\theta_1 < \theta_2$ is that there exist a function $G(z)$, analytic for $|z| < 1$, with $G(0) = 0$ and $|G'(0)| = 1$, for which (9) (with $m = k$) and

$$(20) \quad \Re \left(\frac{f'(z)[f(z)]^m}{z^m G(z)} \right) \geq 0$$

hold.

PROOF. Suppose that (19) holds for all $\theta_1 < \theta_2$, for some real m . Then we apply our lemma, with $p(\theta) = \arg \{f'(e^{i\theta})[f(e^{i\theta})/e^{i\theta}]^m\}$, $m = k$, to obtain the proper $G(z)$ and $F(z) = f'(z) \left[\frac{f(z)}{z} \right]^m$ such that (9), (8) — and hence (20) — hold.

As for the other part of the theorem, it is a simple matter to verify the result. Indeed, this last is one of OGAWA's particular results ([3], pp. 432–436).

Remark 4. OGAWA managed to show that (19) implies that $f(z)$ is univalent; this we have not been able to do by the methods of the present note.

Remark 5. Again, the constant m is not arbitrary; it must at least satisfy the inequality $m > -\frac{1}{2}$.

3. Concluding Remarks

The classes of functions introduced by OGAWA have special properties that should be investigated; distortion inequalities, coefficient inequalities geometric properties, etc. It would also be of interest to determine the complete range of permissible values for the constant m that intervenes. It would be of interest to determine the radius of star-likeness, the radius of convexity, and the radius of "close-to-convexity" for the members of the various classes discussed in this note. Finally, it would be of some interest to determine whether or not each function $f(z)$ that is convex in the direction of the imaginary axis has a representation analogous to (17), and thus obtain some of UMEZAWA's results [4].

References

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