

The extended Hermite—Fejér interpolation formula with application to the theory of generalized almost-step parabolas

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Introduction

Let

$$(1) \quad x_1, x_2, \dots, x_n; \quad \xi_1, \xi_2, \dots, \xi_m$$

be $n+m$ distinct points on the real axis, and let us denote, correspondingly, by

$$(2) \quad y_1, y_2, \dots, y_n; \quad \eta_1, \eta_2, \dots, \eta_m$$

arbitrary real values, further let for the first n points the corresponding real numbers

$$(3) \quad y'_1, y'_2, \dots, y'_n$$

be prescribed as derivative values or *slopes*. The extended Hermite—Fejér interpolation formula (as I wish to call it) yields a polynomial $S(x)$ of degree $2n+m-1$ at most with the property

$$(4) \quad S(x_v) = y_v, \quad S(\xi_k) = \eta_k, \quad S'(x_v) = y'_v \quad (v=1, 2, \dots, n; \quad k=1, 2, \dots, m).$$

As well-known from the theory of interpolation due to CH. HERMITE¹⁾, this polynomial exists always and is unique. We shall find below (§ 1) for it the explicit form (24**). But for this purpose we shall not make use of the general theory.

In particular in the case $y'_1 = \dots = y'_n = 0$ the polynomial $q(x)$ of degree $2n+m-1$ at most with the property

$$(5) \quad q(x_v) = y_v, \quad q(\xi_k) = \eta_k, \quad q'(x_v) = 0 \quad (v=1, 2, \dots, n; \quad k=1, 2, \dots, m)$$

will be called *extended step parabola* belonging to the *fundamental points* under (1) and the ordinates (2). This notion is similar to that of the step parabola due to L. FEJÉR²⁾. Conversely, in the case $y_1 = \dots = y_n = 0, \eta_1 = \dots = \eta_m = 0$ the polynomial $Q(x)$ of degree $2n+m-1$ at most for which

$$(6) \quad Q(x_v) = 0, \quad Q(\xi_k) = 0, \quad Q'(x_v) = y'_v \quad (v=1, 2, \dots, n; \quad k=1, 2, \dots, m)$$

holds, will be named *extended wave parabola* determined by the fundamental points (1) and the slopes (3), corresponding to the notion of wave parabola introduced by L. FEJÉR.³⁾ In general the above polynomial $S(x)$ of degree $2n+m-1$ at most

¹⁾ See [5], 70 or 432, respectively.

²⁾ See [2], 210; further [3], 66.

³⁾ See [4], 714.

characterized by the relations under (4) will be called *generalized extended step parabola* belonging to the fundamental points (1) and the ordinates and slopes under (2), (3) respectively.

In the case $m=2$, $\xi_1=1$, $\xi_2=-1$ the extended Hermite—Fejér interpolation is identical with the quasi-Hermite—Fejér one introduced by the author.⁴⁾ P. TURÁN⁵⁾ proposed that the case $m=1$, $\xi=1$ be investigated also. In §§ 2, 3 of the present paper we deal with this case.

In the just mentioned case $m=1$, $\xi=1$ we shall speak of *almost Hermite—Fejér interpolation formula*. It yields the unique polynomial $S(x)$ of degree $2n$ at most with the property

$$(7) \quad S(x_v) = y_v, \quad S(1) = \eta, \quad S'(x_v) = y'_v \quad (v = 1, 2, \dots, n).$$

In particular in the case $y'_1 = \dots = y'_n = 0$ the polynomial $S(x)$ of degree $2n$ at most characterized by (7) will be called the *almost-step parabola*, while in the case $y_1 = \dots = y_n = 0$, $\eta = 0$ it will be named the *almost-wave parabola*, determined beside the points

$$(8) \quad x_1, x_2, \dots, x_n, 1$$

in the first case by the ordinates

$$(9) \quad y_1, y_2, \dots, y_n, \eta$$

and in the second by the slopes (3). In general it will be called *the generalized almost-step parabola* belonging to the fundamental points (8) and the corresponding ordinates and slopes (9), (3) respectively.

Our Theorem I below (§ 2) states the uniform convergence of the generalized almost-step parabola to the function $f(x)$ continuous in the closed interval $-1 \leq x \leq 1$, in the case when the fundamental points are beside 1 the zeros of the Jacobi polynomial $P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)$ with the notation of G. SZEGŐ⁶⁾ and the ordinates the corresponding values of $f(x)$, provided that the slopes are uniformly bounded if $n \rightarrow +\infty$. Theorem II (§ 3) contains a similar statement for the case of the roots of the Legendre polynomial $P_n(x)$, the convergence however, is then assured only in the interval $-1 < x \leq 1$ open from below, and uniform in each subinterval $-1 + \delta \leq x \leq 1$ ($0 < \delta < 2$).

§ 1. The extended Hermite—Fejér interpolation formula

Using the fundamental points under (1) let us introduce the following notations:

$$(10) \quad \omega(x) = C(x-x_1)(x-x_2)\dots(x-x_n), \quad C \neq 0$$

and

$$(11) \quad \Omega(x) = K(x-\xi_1)(x-\xi_2)\dots(x-\xi_m), \quad K \neq 0$$

⁴⁾ P. SZÁSZ, [7], 414.

⁵⁾ Oral communication for which I am thankful to P. TURÁN.

⁶⁾ See [8], 58.

further

$$(12) \quad l_v(x) = \frac{\omega(x)}{\omega'(x_v)(x - x_v)} \quad (v = 1, 2, \dots, n)$$

and

$$(13) \quad L_k(x) = \frac{\Omega(x)}{\Omega'(\xi_k)(x - \xi_k)} \quad (k = 1, 2, \dots, m).$$

Then we define

$$(14) \quad r_v(x) = \frac{\Omega(x)}{\Omega(x_v)} [1 + c_v(x - x_v)] l_v(x)^2 \quad (v = 1, 2, \dots, n)$$

and

$$(15) \quad \varrho_k(x) = \frac{\omega(x)^2}{\omega(\xi_k)^2} L_k(x) \quad (k = 1, 2, \dots, m)$$

where

$$(16) \quad c_v = \sum_{k=1}^m \frac{1}{\xi_k - x_v} - \frac{\omega''(x_v)}{\omega'(x_v)} \quad (v = 1, 2, \dots, n)$$

to be the *fundamental polynomials of the first kind*, and

$$(17) \quad s_v(x) = \frac{\Omega(x)}{\Omega(x_v)} (x - x_v) l_v(x)^2 \quad (v = 1, 2, \dots, n)$$

to be *those of the second kind*. With regard to (10), (11), (12), (13) it follows by (14), (15), (16), (17) at once

$$(18) \quad r_v(x_v) = 1, \quad r_v(x_\mu) = 0 \quad (\mu \neq v), \quad r_v(\xi_k) = 0 \\ (v, \mu = 1, 2, \dots, n; \quad k = 1, 2, \dots, m)$$

and

$$(19) \quad r'_v(x_\mu) = 0 \quad (v, \mu = 1, 2, \dots, n),$$

further

$$(20) \quad \varrho_k(\xi_k) = 1, \quad \varrho_k(\xi_l) = 0 \quad (l \neq k), \quad \varrho_k(x_v) = 0 \\ (k, l = 1, 2, \dots, m; \quad v = 1, 2, \dots, n)$$

and

$$(21) \quad \varrho'_k(x_v) = 0 \quad (v = 1, 2, \dots, n; \quad k = 1, 2, \dots, m),$$

while

$$(22) \quad s_v(x_\mu) = 0, \quad s_v(\xi_k) = 0 \\ (v, \mu = 1, 2, \dots, n; \quad k = 1, 2, \dots, m)$$

and

$$(23) \quad s'_v(x_v) = 1, \quad s'_v(x_\mu) = 0 \quad (\mu \neq v) \quad (v, \mu = 1, 2, \dots, n).$$

By reason of the relations (18)–(23), if the arbitrary real values under (2) and (3) are prescribed corresponding to the fundamental points (1), the polynomial

$$(24) \quad S(x) = \sum_{v=1}^n y_v r_v(x) + \sum_{k=1}^m \eta_k \varrho_k(x) + \sum_{v=1}^n y'_v s_v(x)$$

of degree $2n + m - 1$ at most has obviously the property (4). Moreover this polynomial $S(x)$ is unique. Indeed, $S^*(x)$ being a polynomial of degree $2n + m - 1$ at most and with the same property, i. e.

$$S^*(x_v) = y_v, \quad S^*(\xi_k) = \eta_k, \quad S^{*'}(x_v) = y'_v \quad (v = 1, 2, \dots, n; \quad k = 1, 2, \dots, m),$$

for the polynomial

$$p(x) = S^*(x) - S(x)$$

of degree $2n + m - 1$ at most there holds

$$\begin{aligned} p(x_v) &= 0, \quad p(\xi_k) = 0, \quad p'(x_v) = 0 \\ (v &= 1, 2, \dots, n; \quad k = 1, 2, \dots, m) \end{aligned}$$

and consequently for all x

$$p(x) = S^*(x) - S(x) = 0,$$

thus $S^*(x)$ is identical with $S(x)$.

As we see, the unique polynomial $S(x)$ of degree $2n + m - 1$ at most with the property (4) is given by (24). Therefore (24) presents the *generalized extended step parabola* defined in the introduction. In particular the *extended step parabola* $q(x)$ characterized by (5) is given by

$$(25) \quad q(x) = \sum_{v=1}^n y_v r_v(x) + \sum_{k=1}^m \eta_k \varrho_k(x)$$

while the *extended wave parabola* $Q(x)$ determined by (6) is

$$(26) \quad Q(x) = \sum_{v=1}^n y'_v s_v(x).$$

Thus, (24) can be written in the abbreviated form

$$(24^*) \quad S(x) = q(x) + Q(x).$$

With respect to (12), (13), (14), (15) we obtain by (25) for $q(x)$ the explicit form

$$(25^*) \quad \begin{aligned} q(x) = & \sum_{v=1}^n y_v \frac{\Omega(x)}{\Omega(x_v)} [1 + c_v(x - x_v)] \left(\frac{\omega(x)}{\omega'(x_v)(x - x_v)} \right)^2 + \\ & + \sum_{k=1}^m \eta_k \frac{\Omega(x)^2}{\Omega(\xi_k)^2} \frac{\Omega(x)}{\Omega'(\xi_k)(x - \xi_k)} \end{aligned}$$

where $\omega(x)$, $\Omega(x)$ and c_v are given by (10), (11) and (16). Further, with regard to (12) and (17), $Q(x)$ can be written by reason of (26) in the detailed form

$$(26^*) \quad Q(x) = \sum_{v=1}^n y'_v \frac{\Omega(x)}{\Omega(x_v)} (x - x_v) \left(\frac{\omega(x)}{\omega'(x_v)(x - x_v)} \right)^2.$$

Consequently, the generalized extended step parabola given by (24*) has the explicit form

$$(24^{**}) \quad S(x) = \sum_{v=1}^n y_v \frac{\Omega(x)}{\Omega(x_v)} [1 + c_v(x - x_v)] \left(\frac{\omega(x)}{\omega'(x_v)(x - x_v)} \right)^2 + \\ + \sum_{k=1}^m \eta_k \frac{\omega(x)^2}{\omega(\xi_k)^2} \frac{\Omega(x)}{\Omega'(\xi_k)(x - \xi_k)} + \sum_{v=1}^n y'_v \frac{\Omega(x)}{\Omega(x_v)} (x - x_v) \left(\frac{\omega(x)}{\omega'(x_v)(x - x_v)} \right)^2$$

where $\omega(x)$, $\Omega(x)$ and c_v are given again by (10), (11), (16). (24) or (24**) is the *extended Hermite—Fejér interpolation formula* as we have named it in the introduction.

In particular, in the case $m=1$, $\xi=1$ we can take $\Omega(x) = 1-x$, and with respect to (12) by (24**) we get

$$(27) \quad S(x) = \sum_{v=1}^n y_v \frac{1-x}{1-x_v} [1 + c_v^*(x - x_v)] l_v(x)^2 + \eta \frac{\omega(x)^2}{\omega(1)^2} + \\ + \sum_{k=1}^m y'_v \frac{1-x}{1-x_v} (x - x_v) l_v(x)^2$$

where

$$(28) \quad c_v^* = \frac{1}{1-x_v} - \frac{\omega''(x_v)}{\omega'(x_v)} \quad (v=1, 2, \dots, n).$$

This is the *almost-Hermite—Fejér interpolation formula* which yields the *generalized almost-step parabola* as we have called it in the introduction.

From the unicity of this generalized almost-step parabola given by (27) it follows that for each polynomial $\Pi(x)$ of degree $2n$ at most

$$\Pi(x) = \sum_{v=1}^n \Pi(x_v) \frac{1-x}{1-x_v} [1 + c_v^*(x - x_v)] l_v(x)^2 + \Pi(1) \frac{\omega(x)^2}{\omega(1)^2} + \\ + \sum_{k=1}^m \Pi'(x_v) (x - x_v) l_v(x)^2$$

holds. This gives for $\Pi(x)=1$ the identity

$$(29) \quad \sum_{v=1}^n \frac{1-x}{1-x_v} [1 + c_v^*(x - x_v)] l_v(x)^2 + \frac{\omega(x)^2}{\omega(1)^2} = 1$$

fundamental in our following treatment; let it be called the *fundamental identity*.

**§ 2. The generalized almost-step parabola corresponding
to the zeros of the Jacobi polynomial $P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)$. Weierstrass
approximation by means of the same**

Disregarding a constant factor the Jacobi polynomial $P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)$ is identical with⁷⁾

$$(30) \quad \omega(x) = \left[\frac{\sin(2n+1)\frac{\vartheta}{2}}{\sin\frac{\vartheta}{2}} \right]_{x=\cos\vartheta}.$$

It has the zeros

$$(31) \quad x_{vn} = \cos \frac{2v\pi}{2n+1} \quad (v=1, 2, \dots, n).$$

To produce the fundamental polynomials of the first and the second kind in the case when $m=1$, $\xi=1$ and the fundamental points beside 1 are those under (31), we start from the well-known differential equation

$$(32) \quad (1-x^2)\omega''(x) - (1+2x)\omega'(x) + n(n+1)\omega(x) = 0.$$

Applying this for the root x_{vn} of $\omega(x)$, we obtain (setting x_v instead of x_{vn})

$$\frac{\omega''(x_v)}{\omega'(x_v)} = \frac{1+2x_v}{1-x_v^2}$$

consequently the coefficient c_v^* under (28) is

$$c_v^* = \frac{1}{1-x_v} - \frac{1+2x_v}{1-x_v^2} = -\frac{x_v}{1-x_v^2}$$

and so we get

$$(33) \quad 1 + c_v^*(x-x_v) = \frac{1-x_vx}{1-x_v^2} \quad (v=1, 2, \dots, n).$$

Thus, by taking $\Omega(x) = 1-x$, the fundamental polynomials of the first kind given by (14) and (15) are

$$(34) \quad \begin{cases} r_v(x) = \frac{1-x}{1-x_v} \frac{1-x_vx}{1-x_v^2} l_v(x)^2 \\ \varrho(x) = \frac{\omega(x)^2}{\omega(1)^2} \end{cases} \quad (v=1, 2, \dots, n),$$

while those of the second kind under (17) are

$$(35) \quad s_v(x) = \frac{1-x}{1-x_v} (x-x_v) l_v(x)^2 \quad (v=1, 2, \dots, n).$$

⁷⁾ G. SZEGÖ, [8], 60, form. (4. 1. 8.).

As we see, in this case of the zeros of the Jacobi polynomial (30) the fundamental polynomials of the first kind under (34) are all non-negative in the interval $-1 \leq x \leq 1$. This fact has a fundamental importance in the following treatment.

With respect to (34) and (35) in the present case the almost-step parabola (25) and the almost-wave parabola (26) are

$$(36) \quad q^*(x) = \sum_{v=1}^n y_v \frac{1-x}{1-x_v} \frac{1-x_v x}{1-x_v^2} l_v(x)^2 + \eta \frac{\omega(x)^2}{\omega(1)^2}$$

and

$$(37) \quad Q^*(x) = \sum_{v=1}^n y'_v \frac{1-x}{1-x_v} (x-x_v) l_v(x)^2,$$

respectively, while the generalized almost-step parabola according to (24*) has the abbreviated form

$$(38) \quad S(x) = q^*(x) + Q^*(x).$$

Now, in what follows we need the following

Lemma I. For the zeros under (31) of the polynomial (30) we have

$$(39) \quad \sum_{v=1}^n \frac{1}{(1-x_{vn})(1-x_{vn}^2)\omega'(x_{vn})^2} = \frac{2n}{(2n+1)^2}.$$

For the proof we start from the identity

$$\sin \frac{\vartheta}{2} \omega(\cos \vartheta) = \sin(2n+1) \frac{\vartheta}{2}.$$

By deriving we get

$$\cos \frac{\vartheta}{2} \omega(\cos \vartheta) - 2 \sin \frac{\vartheta}{2} \sin \vartheta \omega'(\cos \vartheta) = (2n+1) \cos(2n+1) \frac{\vartheta}{2}.$$

Putting here $x = \cos \vartheta$, for the root x_{vn} of $\omega(x)$ there results

$$(40) \quad \omega'(x_{vn}) = (-1)^{v+1} \frac{2n+1}{2 \sin \frac{v\pi}{2n+1} \sin \frac{2v\pi}{2n+1}}.$$

Since with respect to (31)

$$1 - x_{vn} = 2 \sin^2 \frac{v\pi}{2n+1}, \quad 1 - x_{vn}^2 = \sin^2 \frac{2v\pi}{2n+1},$$

by the aid of (40) follows (39), Lemma I is proved.

Next, Weierstrass approximation is expressed by the following⁸⁾

Theorem I. Let $f(x)$ be a continuous function in the closed interval $-1 \leq x \leq 1$.

⁸⁾ First exposed in my lecture „Különböző típusú lépcsőparabolákról” (in Hungarian) on the 6th April 1962 in Debrecen (Hungary), later in the lecture of mine “On generalized quasi-step and almost-step parabolas, respectively” delivered on the 18th August, at the International Congress of Mathematicians Stockholm 1962.

Furthermore let beside 1 the zeros under (31) of the Jacobi polynomial $P_n^{(\frac{1}{2}, -\frac{1}{2})}(x)$ be chosen as fundamental points, and let the arbitrary given numbers

$$y'_{1n}, y'_{2n}, \dots, y'_{nn} \quad (n = 1, 2, 3, \dots)$$

be taken for slopes, provided that

$$(41) \quad |y'_{vn}| \leq \Delta$$

where Δ is independent of n and v . Then for the generalized almost-step parabola $S_n(x)$ of degree $2n$ at most belonging to $f(x)$, i. e. with the property

$$S_n(x_{vn}) = f(x_{vn}), \quad S_n(1) = f(1), \quad S'_n(x_{vn}) = y'_{vn} \quad (v = 1, 2, \dots, n; \quad n = 1, 2, 3, \dots)$$

holds

$$S_n(x) \rightarrow f(x) \quad \text{if } n \rightarrow +\infty$$

the convergence being uniform in the whole closed interval.

PROOF. We prove first of all that the almost-step parabola $q_n^*(x)$ of degree $2n$ at most corresponding to $f(x)$, that is according to (36)

$$(42) \quad q_n^*(x) = \sum_{v=1}^n f(x_{vn}) \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 + f(1) \frac{\omega_n(x)^2}{\omega_n(1)^2}$$

where $\omega_n(x)$ is identical with the polynomial $\omega(x)$ under (30), and by (12)

$$(43) \quad l_{vn}(x) = \frac{\omega_n(x)}{\omega'_n(x_{vn})(x-x_{vn})},$$

converges uniformly to $f(x)$ if $n \rightarrow +\infty$.

With regard to (33), in the present case the fundamental identity (29) has the form

$$(44) \quad \sum_{v=1}^n \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 + \frac{\omega_n(x)^2}{\omega_n(1)^2} = 1.$$

Consequently it may be written

$$(45) \quad f(x) = \sum_{v=1}^n f(x_{vn}) \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 + f(x) \frac{\omega_n(x)^2}{\omega_n(1)^2}$$

and by (42), (45) there follows

$$(46) \quad f(x) - q_n^*(x) = \sum_{v=1}^n [f(x) - f(x_{vn})] \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 + [f(x) - f(1)] \frac{\omega_n(x)^2}{\omega_n(1)^2}.$$

But

$$[f(x) - f(1)] \frac{\omega_n(x)^2}{\omega_n(1)^2} \rightarrow 0$$

uniformly for $-1 \leq x \leq 1$. This is an easy consequence of the fact that the first

factor [...] vanishes for $x = 1$ and is continuous for $-1 \leq x \leq 1$, while with regard to (30)

$$\frac{\omega_n(x)}{\omega_n(1)} = \frac{1}{2n+1} \left[\frac{\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} \right]_{x=\cos\theta} \rightarrow 0$$

uniformly in each subinterval $-1 \leq x \leq 1 - \sigma$ ($0 < \sigma < 2$) and is bounded in the whole closed interval since as well-known

$$\left| \frac{\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} \right| \leq 2n+1.$$

Therefore (46) implies the uniform convergence of $q_n^*(x)$ to $f(x)$ if one shows that

$$(47) \quad \sum_{v=1}^n [f(x) - f(x_{vn})] \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 \rightarrow 0$$

uniformly for $-1 \leq x \leq 1$. This may be seen as follows.

By reason of the theorem of Weierstrass let for $-1 \leq x \leq 1$ be

$$(48) \quad |f(x)| \leq M.$$

The positive number ε having been chosen as small as we please, there exists a positive number δ such that

$$(49) \quad |f(x') - f(x'')| \leq \varepsilon \quad \text{when} \quad |x' - x''| \leq \delta,$$

the continuous function $f(x)$ being, as well-known, also uniformly continuous for $-1 \leq x \leq 1$. Owing to the non-negativity of the fundamental polynomials under (34), by the aid of (44), (49) we have for a fixed x in the interval $-1 \leq x \leq 1$

$$(50) \quad \left| \sum_{|x_{vn}-x| \leq \delta} [f(x) - f(x_{vn})] \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 \right| \leq \\ \leq \varepsilon \sum_{|x_{vn}-x| \leq \delta} \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 \leq \varepsilon.$$

On the other hand, in view of (48), (43) and the obvious inequality $0 < 1 - x_{vn}x < 2$ we have

$$(51) \quad \left| \sum_{|x_{vn}-x| > \delta} [f(x) - f(x_{vn})] \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 \right| \leq \\ \leq \frac{4M}{\delta^2} \sum_{|x_{vn}-x| > \delta} \frac{(1-x)\omega_n(x)^2}{(1-x_{vn})(1-x_{vn}^2)\omega_n'(x_{vn})^2}.$$

However, with regard to (30) it is clear that

$$(52) \quad 0 \leq (1-x)\omega_n(x)^2 \leq 2 \quad \text{for } -1 \leq x \leq 1.$$

Hence by the aid of (51) and of Lemma I under (39) there follows that

$$(53) \quad \left| \sum_{|x_{vn}-x|>\delta} [f(x)-f(x_{vn})] \frac{1-x}{1-x_{vn}^2} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 \right| \leq \frac{8M}{\delta^2} \frac{2n}{(2n+1)^2} \leq \varepsilon$$

if n is large enough, i. e. $n \geq N$, say. Now, by (50) and (53) we have for $-1 \leq x \leq 1$

$$\left| \sum_{v=1}^n [f(x)-f(x_{vn})] \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 \right| \leq 2\varepsilon$$

when $n \geq N$, i. e. (47) is valid. Thus, the uniform convergence of the almost-step parabola (42) to $f(x)$ is established.

The generalized almost-step parabola $S_n(x)$ being by (37) and (38)

$$S_n(x) = q_n^*(x) + \sum_{v=1}^n y'_{vn} \frac{1-x}{1-x_{vn}} (x-x_{vn}) l_{vn}(x)^2,$$

for the proof of the present theorem there remains still to show that

$$\sum_{v=1}^n y'_{vn} \frac{1-x}{1-x_{vn}} (x-x_{vn}) l_{vn}(x)^2 \rightarrow 0$$

uniformly for $-1 \leq x \leq 1$ if $n \rightarrow +\infty$. But by the assumption (41) we have

$$\left| \sum_{v=1}^n y'_{vn} \frac{1-x}{1-x_{vn}} (x-x_{vn}) l_{vn}(x)^2 \right| \leq \Delta \sum_{v=1}^n \frac{1-x}{1-x_{vn}} |x-x_{vn}| l_{vn}(x)^2.$$

Hence, for our purpose it is sufficient to show that

$$(54) \quad \sum_{v=1}^n \frac{1-x}{1-x_{vn}} |x-x_{vn}| l_{vn}(x)^2 \rightarrow 0$$

uniformly for $-1 \leq x \leq 1$.

First, the identity (44) implies

$$(55) \quad \sum_{v=1}^n \frac{1-x}{1-x_{vn}} \frac{1-x_{vn}x}{1-x_{vn}^2} l_{vn}(x)^2 \leq 1.$$

However, by the obvious inequalities

$$0 < 1+x_{vn} < 2, \quad 0 < 1-x_{vn} < 2,$$

one has

$$\left[\frac{1-x_{vn}x}{1-x_{vn}^2} \right]_{x=1} = \frac{1}{1+x_{vn}} > \frac{1}{2}, \quad \left[\frac{1-x_{vn}x}{1-x_{vn}^2} \right]_{x=-1} = \frac{1}{1-x_{vn}} > \frac{1}{2},$$

consequently

$$\frac{1-x_{vn}x}{1-x_{vn}^2} > \frac{1}{2} \quad \text{for } -1 \leq x \leq 1,$$

and so by reason of (55) we have

$$(56) \quad \sum_{v=1}^n \frac{1-x}{1-x_{vn}} l_{vn}(x)^2 \leq 2 \quad (-1 \leq x \leq 1)$$

in view to the non-negativity of the terms. Now, let an arbitrary positive number ε be given. Owing to the inequality (56) we have for a fixed x

$$(57) \quad \sum_{|x_{vn}-x| \leq \varepsilon} \frac{1-x}{1-x_{vn}} |x-x_{vn}| l_{vn}(x)^2 \leq \varepsilon \quad \sum_{|x_{vn}-x| \leq \varepsilon} \frac{1-x}{1-x_{vn}} l_{vn}(x)^2 \leq 2\varepsilon.$$

On the other hand, in view of (43), (52) and the obvious inequality $0 < 1-x_{vn}^2 < 1$, by Lemma I under (39) there follows

$$\begin{aligned} \sum_{|x_{vn}-x| > \varepsilon} \frac{1-x}{1-x_{vn}} |x-x_{vn}| l_{vn}(x)^2 &\leq 2 \sum_{|x_{vn}-x| > \varepsilon} \frac{1}{(1-x_{vn}) \omega'_n(x_{vn})^2 |x-x_{vn}|} < \\ &< \frac{2}{\varepsilon} \sum_{|x_{vn}-x| > \varepsilon} \frac{1}{(1-x_{vn})(1-x_{vn}^2) \omega'_n(x_{vn})^2} \leq \frac{2}{\varepsilon} \frac{2n}{(2n+1)^2}. \end{aligned}$$

Consequently,

$$(58) \quad \sum_{|x_{vn}-x| > \varepsilon} \frac{1-x}{1-x_{vn}} |x-x_{vn}| l_{vn}(x)^2 < \varepsilon$$

when n is large enough, i. e. $n \geq N'$, say. Combining (57) and (58) we obtain for $-1 \leq x \leq 1$

$$\sum_{v=1}^n \frac{1-x}{1-x_{vn}} |x-x_{vn}| l_{vn}(x)^2 < 3\varepsilon \quad \text{if only} \quad n \geq N'$$

which expresses just the validity of (54), thus completing the proof of the theorem.

§ 3. Weierstrass approximation in each subinterval $-1+\delta \leq x \leq 1$ ($1 < \delta < 2$) by means of the generalized almost-step parabola belonging to the zeros of the Legendre polynomial $P_n(x)$

The Legendre polynomial $\omega(x) = P_n(x)$ of degree n (exactly) satisfies the differential equation

$$(1-x^2)\omega''(x) - 2x\omega'(x) + n(n+1)\omega(x) = 0;$$

for the zeros

$$(59) \quad x_{1n}, x_{2n}, \dots, x_{nn}$$

of the same we have (setting x_v instead of x_{vn})

$$\frac{\omega''(x_v)}{\omega'(x_v)} = \frac{2x_v}{1-x_v^2} \quad (v=1, 2, \dots, n).$$

Consequently, in the present case for the coefficient c_v^* by (28) results

$$c_v^* = \frac{1}{1-x_v} - \frac{2x_v}{1-x_v^2} = \frac{1}{1+x_v}$$

and so we obtain

$$(60) \quad 1 + c_v^*(x - x_v) = \frac{1 + x}{1 + x_v} \quad (v = 1, 2, \dots, n).$$

Thus, with respect to $P_n(1) = 1$, the fundamental polynomials of the first kind under (14) and (15) are (taking $\Omega(x) = 1 - x$ again)

$$(61) \quad \begin{cases} r_v(x) = \frac{1-x}{1-x_v} \frac{1+x}{1+x_v} l_v(x)^2 = \frac{1-x^2}{1-x_v^2} \left(\frac{P_n(x)}{P'_n(x_v)(x-x_v)} \right)^2 \\ (v = 1, 2, \dots, n), \\ q(x) = P_n(x)^2, \end{cases}$$

while those of the second kind given by (17) are

$$(62) \quad s_v(x) = \frac{1-x}{1-x_v} (x - x_v) l_v(x)^2 = \frac{1-x}{1-x_v} (x - x_v) \left(\frac{P_n(x)}{P'_n(x_v)(x-x_v)} \right)^2 \\ (v = 1, 2, \dots, n).$$

As we see, also in this case $\omega(x) = P_n(x)$ the fundamental polynomials of the first kind are all non-negative in the interval $-1 \leq x \leq 1$. This fact is again of fundamental importance in what follows.

With regard to (61), in the present case the fundamental identity (29) has the form

$$(63) \quad \sum_{v=1}^n \frac{1-x^2}{1-x_v^2} \left(\frac{P_n(x)}{P'_n(x_v)(x-x_v)} \right)^2 + P_n(x)^2 = 1$$

already discovered by E. EGÉRVÁRY and P. TURÁN⁹⁾ but in another way.

By the aid of (61) and (62) the almost-step parabola (25*) and the almost-wave parabola (26*) have at present the forms

$$(64) \quad q^*(x) = \sum_{v=1}^n y_v \frac{1-x^2}{1-x_v^2} \left(\frac{P_n(x)}{P'_n(x_v)(x-x_v)} \right)^2 + \eta P_n(x)^2$$

and

$$(65) \quad Q^*(x) = \sum_{v=1}^n y'_v \frac{1-x}{1-x_v} (x - x_v) \left(\frac{P_n(x)}{P'_n(x_v)(x-x_v)} \right)^2,$$

respectively.

Now, in what follows we need the following

Lemma II. For the roots under (59) of the Legendre polynomial $P_n(x)$ we have

$$(66) \quad \sum_{v=1}^n \frac{1-x^2}{1-x_{vn}^2} |x - x_{vn}| \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 \rightarrow 0$$

uniformly for $-1 \leq x \leq 1$.

⁹⁾ See [1], 264, form. (6. 7).

This fact was already proved by the author in the paper quoted above.¹⁰⁾ For the sake of completeness we repeat here the proof with a slight modification.

Let the positive number ε be chosen as small as we please. Owing to (63) and the non-negativity of the fundamental polynomials of the first kind under (61), we have for fixed x

$$(67) \quad \sum_{|x_{vn}-x| \leq \varepsilon} \frac{1-x^2}{1-x_{vn}^2} |x-x_{vn}| \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 \leq \\ \leq \varepsilon \sum_{|x_{vn}-x| \leq \varepsilon} \frac{1-x^2}{1-x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 \leq \varepsilon.$$

On the other hand, by the aid of the relation

$$(68) \quad \sum_{v=1}^n \frac{1}{(1-x_{vn}^2) P'_n(x_{vn})^2} = 1$$

shown by L. FEJÉR¹¹⁾

$$\sum_{|x_{vn}-x| > \varepsilon} \frac{1-x^2}{1-x_{vn}^2} |x-x_{vn}| \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 = \\ = (1-x^2) P_n(x)^2 \sum_{|x_{vn}-x| > \varepsilon} \frac{1}{(1-x_{vn}^2) P'_n(x_{vn})^2} \frac{1}{|x-x_{vn}|} \leq \frac{1}{\varepsilon} (1-x^2) P_n(x)^2$$

holds. Thus, taking into account T. J. STIELTJES's estimation¹²⁾

$$(69) \quad |P_n(x)| < \frac{c}{\sqrt[n]{n}} \frac{1}{\sqrt[4]{1-x^2}} \quad \text{for} \quad -1 < x < 1$$

where c is a numerical constant, we have obviously

$$(70) \quad \sum_{|x_{vn}-x| > \varepsilon} \frac{1-x^2}{1-x_{vn}^2} |x-x_{vn}| \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 \leq \varepsilon$$

if n is large enough, i. e. $n \geq N$, say. Combining (67) and (70) there follows for $-1 \leq x \leq 1$

$$\sum_{v=1}^n \frac{1-x^2}{1-x_{vn}^2} |x-x_{vn}| \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 \leq 2\varepsilon \quad \text{if only} \quad n \geq N,$$

i. e. (66) is valid, Lemma II is proved.

Next, similarly to our Theorem I (§ 2) Weierstrass approximation is expressed, but only in each subinterval $-1+\delta \leq x \leq 1$ ($0 < \delta < 2$) by the following

Theorem II. Let $f(x)$ be a continuous function in the closed interval $-1 \leq x \leq 1$. Furthermore let beside 1 the zeros under (59) of the Legendre polynomial $P_n(x)$ be chosen as fundamental points, and let the arbitrary given numbers

$$y'_{1n}, y'_{2n}, \dots, y'_{nn} \quad (n = 1, 2, 3, \dots)$$

¹⁰⁾ P. SZÁSZ, [7], 424.

¹¹⁾ See [2], 221, form. (45); or [3], 78, form. (39).

¹²⁾ See [6], 241–242.

be taken for slopes, provided that

$$(71) \quad |y'_{vn}| \leq \Delta \quad (v = 1, 2, \dots, n; n = 1, 2, 3, \dots)$$

Δ being independent of n and v . Then for the generalized almost-step parabola $S_n(x)$ of degree $2n$ at most corresponding to $f(x)$, i. e. with the property

$$S_n(x_{vn}) = f(x_{vn}), \quad S_n(1) = f(1), \quad S'_n(x_{vn}) = y'_{vn} \\ (v = 1, 2, \dots, n; n = 1, 2, 3, \dots)$$

the relation

$$S_n(x) \rightarrow f(x) \text{ for } -1 < x \leq 1$$

holds if $n \rightarrow +\infty$, the convergence being uniform in each subinterval $-1 + \delta \leq x \leq 1$ ($0 < \delta < 2$).

PROOF. First we prove that the almost-step parabola $q_n^*(x)$ of degree $2n$ at most corresponding to $f(x)$, that is by (64)

$$(72) \quad q_n^*(x) = \sum_{v=1}^n f(x_{vn}) \frac{1-x^2}{1-x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 + f(1) P_n(x)^2$$

converges uniformly to $f(x)$ for $-1 + \delta \leq x \leq 1$ ($0 < \delta < 2$) if $n \rightarrow +\infty$.

By the aid of (63) it may be written

$$(73) \quad f(x) = \sum_{v=1}^n f(x) \frac{1-x^2}{1-x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 + f(x) P_n(x)^2$$

and by (72), (73) follows

$$(74) \quad f(x) - q_n^*(x) = \\ = \sum_{v=1}^n [f(x) - f(x_{vn})] \frac{1-x^2}{1-x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 + [f(x) - f(1)] P_n(x)^2.$$

But

$$(75) \quad [f(x) - f(1)] P_n(x)^2 \rightarrow 0$$

uniformly for $-1 + \delta \leq x \leq 1$ ($0 < \delta < 2$). This is an easy consequence of the fact that the first factor [...] vanishes for $x=1$ and is continuous for $-1 \leq x \leq 1$, while $P_n(x) \rightarrow 0$ uniformly in each internal subinterval $-1 + \sigma \leq x \leq 1 - \sigma$ ($0 < \sigma < 1$) and is bounded in the whole closed interval. Hence it is sufficient to prove that

$$(76) \quad \sum_{v=1}^n [f(x) - f(x_{vn})] \frac{1-x^2}{1-x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 \rightarrow 0$$

uniformly¹³⁾ for $-1 \leq x \leq 1$ if $n \rightarrow +\infty$.

¹³⁾ This fact was already shown by E. EGÉRVÁRY and P. TURÁN ([1], 264–265) and later a little differently by the author ([7], 422–423). For the sake of completeness we repeat the proof in the text.

By reason of the theorem of Weierstrass there exists a positive number M for which

$$(77) \quad |f(x)| \leq M \quad (-1 \leq x \leq 1).$$

Further the continuous function $f(x)$ is also uniformly continuous in the interval $-1 \leq x \leq 1$, therefore the positive number ε having been chosen as small as we please, there exists a positive number τ such that

$$(78) \quad |f(x') - f(x'')| \leq \varepsilon \quad \text{when} \quad |x' - x''| < \tau.$$

By the aid of (78), (63) and the non-negativity of the fundamental polynomials of the first kind under (61), we have for fixed x

$$(79) \quad \left| \sum_{|x_{vn} - x| \leq \tau} [f(x) - f(x_{vn})] \frac{1 - x^2}{1 - x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x - x_{vn})} \right)^2 \right| \leq \varepsilon \sum_{|x_{vn} - x| \leq \tau} \frac{1 - x^2}{1 - x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x - x_{vn})} \right)^2 \leq \varepsilon.$$

On the other hand, by (77) and (68)

$$(80) \quad \left| \sum_{|x_{vn} - x| > \tau} [f(x) - f(x_{vn})] \frac{1 - x^2}{1 - x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x - x_{vn})} \right)^2 \right| \leq \frac{2M}{\tau^2} (1 - x^2) P_n(x)^2 \sum_{|x_{vn} - x| > \tau} \frac{1}{(1 - x_{vn}^2) P'_n(x_{vn})^2} \leq \frac{2M}{\tau^2} (1 - x^2) P_n(x)^2$$

holds. Taking (69) into account again, we obtain by (80) at once

$$(81) \quad \left| \sum_{|x_{vn} - x| > \tau} [f(x) - f(x_{vn})] \frac{1 - x^2}{1 - x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x - x_{vn})} \right)^2 \right| \leq \varepsilon$$

if n is large enough, i. e. $n \geq N'$, say. Finally, by (79), (81) there follows

$$\left| \sum_{v=1}^n [f(x) - f(x_{vn})] \frac{1 - x^2}{1 - x_{vn}^2} \left(\frac{P_n(x)}{P'_n(x_{vn})(x - x_{vn})} \right)^2 \right| \leq 2\varepsilon$$

for $-1 \leq x \leq 1$ if $n \geq N'$. Thus, (76) is established, i. e. the almost-step parabola (72) converges uniformly to $f(x)$, for $-1 + \delta \leq x \leq 1$ if $n \rightarrow +\infty$.

Now, since the generalized almost-step parabola $S_n(x)$ is according to (38) the sum of $q_n^*(x)$ and $Q_n^*(x)$, that is by (65)

$$S_n(x) = q_n^*(x) + \sum_{v=1}^n y'_{vn} \frac{1 - x}{1 - x_{vn}} (x - x_{vn}) \left(\frac{P_n(x)}{P'_n(x_{vn})(x - x_{vn})} \right)^2,$$

for a proof of our theorem there remains to be shown that

$$(82) \quad \sum_{v=1}^n y'_{vn} \frac{1 - x}{1 - x_{vn}} (x - x_{vn}) \left(\frac{P_n(x)}{P'_n(x_{vn})(x - x_{vn})} \right)^2 \rightarrow 0$$

uniformly for $-1 + \delta \leq x \leq 1$. This may be seen as follows.

Since

$$y'_{vn} \frac{1-x}{1-x_{vn}} = y'_{vn} \frac{1+x_{vn}}{1+x} \frac{1-x^2}{1-x_{vn}^2},$$

by reason of the assumption under (71) and the obvious inequality

$$\left| \frac{1+x_{vn}}{1+x} \right| < \frac{2}{\delta} \quad \text{for} \quad -1+\delta \leq x \leq 1,$$

we have in the same interval

$$\begin{aligned} & \left| \sum_{v=1}^n y'_{vn} \frac{1-x}{1-x_{vn}} (x-x_{vn}) \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2 \right| \leq \\ & \leq \frac{2A}{\delta} \sum_{v=1}^n \frac{1-x^2}{1-x_{vn}^2} |x-x_{vn}| \left(\frac{P_n(x)}{P'_n(x_{vn})(x-x_{vn})} \right)^2. \end{aligned}$$

Thus, from Lemma II under (66) results (82), with which the proof of the theorem is complete.

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