

A note on left regular semigroups

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Abstract. In this paper we define left completely simple semigroups and investigate new decompositions of left regular semigroups.

Various concepts of regularity have been investigated by R. CROISOT in [5], and his study have been presented in the book of A. H. CLIFFORD and G. B. PRESTON [3], as Croisot's theory. One of the central places in this theory is held by the left regularity. Among the various characterizations of left regular semigroups given in Theorem 4.2 [3], we meet a characterization by decompositions into a union of left simple semigroups. Decompositions of these semigroups into left simple components were also a topic of investigations of S. BOGDANOVIĆ [1], M. PETRICH [6,7] and P. PROTIC [9], who studied right zero bands, semilattices and bands of left simple semigroups.

Here we consider some other aspects of decompositions of left regular semigroups. We introduce the notion of left completely simple semigroup, which is a generalization of the notion of completely simple semigroup, we give various characterizations of these semigroups and we characterize left regular semigroups as semilattices of left completely simple semigroups. We also study various band decompositions with left simple components.

Throughout this paper, $\mathbb{Z}$ will denote the set of positive integers, and for a semigroup S , $\text{Intra}(S)$ and $\text{LReg}(S)$ will denote the set of intra-regular and the set of left regular elements of S , respectively, and the relations $|$ and $\underset{l}{|}$ on S will be defined by

$$a | b \iff (\exists x, y \in S^1) b = xay, \quad a \underset{l}{|} b \iff (\exists x \in S^1) b = xa, \quad (a, b \in S).$$

A semigroup S will be called *intra- π -regular* (*left π -regular*) if for any element a of S , some power of a is intra-regular (left regular). In the sense of terminology from [8], the notion “matrix of semigroups” means “rectangular band of semigroups”. A partially ordered set is *discrete* if no two its distinct elements comparable.

For undefined notions and notations we refer to [2], [3], [4], [7] and [8].

We start with the following

Theorem 1. *A semigroup S is left π -regular if and only if it is intra- π -regular and $\text{Intra}(S) = \text{LReg}(S)$.*

PROOF. Let S be left π -regular. Clearly, S is intra π -regular and $\text{LReg}(S) \subseteq \text{Intra}(S)$. Assume $a \in \text{Intra}(S)$. Then $a = xa^2y$, for some $x, y \in S$, whence $a = (xa)^n ay^n$, for each $n \in \mathbb{Z}^+$. Since S is left π -regular, then $(xa)^n = z(xa)^{2n}$, for some $n \in \mathbb{Z}^+$, $z \in S$, whence

$$a = (xa)^n ay^n = z(xa)^{2n} ay^n = z(xa)^n a \in Sa^2.$$

Therefore, $a \in \text{LReg}(S)$, so $\text{Intra}(S) = \text{LReg}(S)$.

The converse is immediate. □

A semigroup S will be called *left (right) completely simple* if it is simple and left (right) regular. It is well-known that a semigroup S is completely simple if and only if it is simple and completely regular, whence we have that S is completely simple if and only if it is both left and right completely simple.

Now we will characterize left completely simple semigroups. For some related results see A. H. CLIFFORD and G. B. PRESTON [4, Theorem 6.36] and S. BOGDANOVIĆ [1, Theorem 2.3].

Theorem 2. *The following conditions on a semigroup S are equivalent:*

- (i) S is left completely simple;
- (ii) S is simple and left π -regular;
- (iii) S is a matrix of left simple semigroups;
- (iv) S is a right zero band of left simple semigroups;
- (v) $(\forall a, b \in S) a \in Sba$;
- (vi) $|$ is a symmetric relation on S ;
- (vii) $S/\mathcal{L}$ is a discrete partially ordered set.

PROOF. (iv) $\iff$ (v). This have been proved in Theorem 2.3 of [1].

(i) $\implies$ (ii). This is obvious.

(ii) $\implies$ (i). Since S is simple, then $S = \text{Intra}(S)$. Now by Theorem 1 we obtain that $S = \text{Intra}(S) = \text{LReg}(S)$, so S is left regular.

(iii) $\implies$ (iv). If S is a matrix of left simple semigroups, then it is a right zero band of semigroups that are left zero bands of left simple semigroups. Since a left zero band of left simple semigroups is also a left simple semigroup, then we obtain (iv).

(iv) $\implies$ (iii). This is clear.

(i) $\implies$ (v). For $a, b \in S$ we have that $a = xby$, for some $x, y \in S^1$, and $xb = z(xb)^2$, for some $z \in S$, whence

$$a = xby = z(xb)^2y = zxb(xby) = zxba \in Sba.$$

(v) $\implies$ (i). This is immediate.

(iv) $\implies$ (vi). Let S be a right zero band I of left simple semigroups S_i , $i \in I$. Assume $a, b \in S$ such that $a \mid b$, i.e. $b = xa$, for some $x \in S^1$. Then $a, b \in S_i$, for some $i \in I$, and S_i is left simple, whence $b \mid a$.

(vi) $\implies$ (v). For all $a, b \in S$, $a \mid ba$, and by the hypothesis, $ba \mid a$, i.e. $a \in S^1ba$, which yields $a \in Sba$.

(vi) $\implies$ (vii). Assume $L_a, L_b \in S/\mathcal{L}$ such that $L_a \leq L_b$, i.e. such that $a \in S^1b$. Then $b \mid a$, so by (vi) we obtain that $a \mid b$, i.e. $b \in S^1a$, whence $L_b \leq L_a$. Thus, $L_a = L_b$. This proves (vii).

(vii) $\implies$ (vi). Assume $a, b \in S$ such that $a \mid b$. Then $L_b \leq L_a$, and by (vii) it follws that $L_b = L_a$, whence $b \mid a$. Hence, $\mid$ is symmetric. $\square$

Note that several known characterizations of completely simple semigroups can be obtained immediately by the previous theorem and its dual.

As we noted above, various characterizations of left regular semigroups were given by R. CROISOT in [5] and in the book of A. H. CLIFFORD and G. B. PRESTON [4]. Here we give some new characterizations of these semigroups.

Theorem 3. *The following conditions on a semigroup S are equivalent:*

- (i) S is left regular;
- (ii) S is intra-regular and left π -regular;
- (iii) S is a semilattice of left completely simple semigroups;
- (iv) S is a union of left completely simple semigroups;
- (v) S is a semilattice of right zero bands of left simple semigroups;
- (vi) $(\forall a, b \in S) a \mid b \implies ab \mid b$.

PROOF. (i) $\implies$ (ii). This is clear.

(ii) $\implies$ (iii). By the well-known Croisot's theorem (Theorem 4.4 of [3]), S is a semilattice of simple semigroups S_α , $\alpha \in Y$. For any $\alpha \in Y$, S_α is also left π -regular, so by Theorem 2, it is left completely simple.

(iii) $\implies$ (vi). Assume $a, b \in S$ such that $a \mid b$. By the hypothesis, there exists a left completely simple subsemigroup A of S such that $b, ba \in A$, and by Theorem 2, $b \in Abab \subseteq Sab$.

(vi) $\implies$ (i). This is obvious.

(iii) $\iff$ (iv). This follows from Theorem 2 and Theorem 4.2 of [3].

(iii) $\iff$ (v). This follows immediately from Theorem 2. $\square$

As consequences of Theorem 3 and its dual we can obtain various known results concerning completely regular semigroups (see [2], [3] and [7]).

Bands of left simple semigroups were described by P. PROTIC [9]. We will give a more simple characterization of such semigroups.

Theorem 4. *A semigroup S is a band of left simple semigroups if and only if it is left regular and $ab \in Sab^2$, for all $a, b \in S$.*

PROOF. Let ϱ denote the greatest congruence relation on S contained in $\mathcal{L}$. Since $\mathcal{L}$ is a right congruence, then by Lemma 10.3 of [4] we have that

$$\varrho = \{(a, b) \in S \times S \mid (\forall x \in S^1) (xa, xb) \in \mathcal{L}\}.$$

Now, S is left regular and $ab \in Sab^2$, for all $a, b \in S$, if and only if ϱ is a band congruence on S .

On the other hand, for any band congruence θ on S , each θ -class is a left simple semigroup if and only if $\theta \subseteq \mathcal{L}$. Indeed, assume that each θ -class of S is simple and let $a \theta b$. Then a and b are contained in the same θ -class A of S . Since A is left simple, $ax = b$ and $b = ya$, for some $x, y \in A^1$,

whence $a \mathcal{L} b$. Conversely, assume $\theta \subseteq \mathcal{L}$ and let a and b be elements of a θ -class A of S . Since $a^2 \in A$, we have $(a^2, b) \in \theta \subseteq \mathcal{L}$, whence $b = xa^2$ for some $x \in S^1$. Moreover, it follows from $a \theta a^2$ that $xa \theta xa^2 = b$, and hence $xa \in A$. Then $b = xa^2 = (xa)a \in Aa$. Thus A is left simple.

Applying the above result on ϱ we obtain the assertion of the theorem. $\square$

By the previous theorem and its dual we can obtain the known characterization of bands of groups given in the book of M. PETRICH [7].

Further we consider some special types of band decompositions with left simple components.

Theorem 5. *The following conditions on a semigroup S are equivalent:*

- (i) S is a left semiregular band of left simple semigroups;
- (ii) S is a right regular band of left simple semigroups;
- (iii) $\mathcal{L}$ is a band congruence on S ;
- (iv) S is left regular and $axy \in Sayxy$, for all $a, x, y \in S$.

PROOF. (i) $\implies$ (ii). By the dual of Proposition II.3.5 of [8], any left semiregular band is a right regular band of left zero bands. Now, if S is a left semiregular band of left simple semigroups, then it is a right regular band of semigroups that are left zero bands of left simple semigroups. As we noted above, a left zero band of left simple semigroups is also left simple, whence S is a right regular band of left simple semigroups.

(ii) $\implies$ (iv). Let S be a right regular band of left simple semigroups and let θ be the related band congruence on S . Clearly, S is left regular. Assume $a, x, y \in S$. Since S/θ is a right regular band, then $xy \theta yxy$, whence $axy \theta ayxy$. If A denotes the θ -class containing $ayxy$ and axy , then $axy \in Aayxy \subseteq Sayxy$, since A is left simple.

(iv) $\implies$ (iii). By Theorem 3, S is a semilattice of left completely simple semigroups. For all $a, x, y \in S$, axy and $ayxy$ are in the same component of this semilattice decomposition. Since by Theorem 2, $|$ is a symmetric relation on this component, then by the hypothesis we obtain that

$$(1) \quad axy \mathcal{L} ayxy, \quad \text{for all } a, x, y \in S.$$

On the other hand, for all $a, b \in S$, $b = xb$, for some $x \in S$, since S is left regular, so $ab = axb \mathcal{L} abxb = ab^2$. Therefore,

$$(2) \quad ab \mathcal{L} ab^2, \quad \text{for all } a, b \in S.$$

Now, assume $a, b \in S$ such that $a \mathcal{L} b$ and assume $x \in S$. Then $b = ua$, $a = vb$, for some $u, v \in S$, so by (1) and (2) we obtain

$$xa = xvb \mathcal{L} xvb^2 = xab \quad \text{and} \quad xb = xua \mathcal{L} xua^2 = xba$$

whence it follows that $xb^2 \mathcal{L} xbab \mathcal{L} xab \mathcal{L} xa$, so that $xb \mathcal{L} xa$. Therefore, $\mathcal{L}$ is a congruence on S .

Since S is left regular, then $\mathcal{L}$ is a band congruence.

(iii) $\implies$ (i). Assume arbitrary $a, x, y \in S$. Then $xy \mathcal{L} yxy$, whence $axy \mathcal{L} ayxy$ and $axy \mathcal{L} (axy)^2 \mathcal{L} axyayxy$. Thus, $S/\mathcal{L}$ is a left semiregular band. By the proof of Theorem 4, any $\mathcal{L}$ -class is a left simple semigroup. $\square$

In a similar way we can prove the following:

Corollary 1. *The following conditions on a semigroup S are equivalent:*

- (i) S is a left seminormal band of left simple semigroups;
- (ii) S is a right normal band of left simple semigroups;
- (iii) S is left regular and $ayx \in Saxy$, for all $a, x, y \in S$.

Corollary 2. *The following conditions on a semigroup S are equivalent:*

- (i) S is a left regular band of left simple semigroups;
- (ii) S is a semilattice of left simple semigroups;
- (iii) S is left regular and $ab \in Sa$, for all $a, b \in S$.

Note that (ii) $\iff$ (iii) of Corollary 1 has been proved by P. PROTIĆ in [9], and (ii) $\iff$ (iii) of Corollary 2 has been proved by M. PETRICH in [6] (see also [7]).

Acknowledgement. We thank the referees for their valuable suggestions and comments. Also we are indebted to Professor P. PROTIĆ for letting us see his preprint [9].

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(Received March 17, 1995; revised September 15, 1995)