

On unitary and Π -unitary rings

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We call an element a of a ring A right (left) unitary if and only if for each a in A , there exists an element $x = x(a)$ of A such that $a = ax$ (or $a = xa$) and a ring is right (left) unitary if and only if each element of A is right (left) unitary. Every right (left) valuation ring is right (left) unitary but not conversely. Further, an element a of a ring is said to be Π -unitary if and only if there exists a positive integer $n = n(a)$ and an element $x = x(a)$ in the center such that $a^n = a^n x$. A ring all of whose elements are Π -unitary is termed as Π -unitary.

We have obtained a few results about such rings and also prove a theorem on the existence of a unity. The arguments are very straight forward and do not depend much on the structure theorems. We shall refer to a subring C of a ring A (cf. [2]) defined as follows:

$$C = \{c | c \in A; ce = ec = c\}$$

where e is some fixed element of A .

1. We begin with

Theorem 1. *In a right (left) unitary ring A if to each $a \in A$, there corresponds a positive integer $n = n(a)$ such that $a^n \in C$, then A is either nil or it has an identity element.*

PROOF. Let $a \in A$. Then $a^{n(a)} \in C$. In particular, $e^n \in C$. It follows then, $e^n e = e^n$, viz., $e^{n+1} = e^n$. In general $e^{n+k} = e^n$ ($k = 0, 1, 2, \dots$). This implies that $w = e^n$ is an idempotent. It is easy to verify that $wAw = C$. In case $w = 0$, $C = 0$. Therefore A is nil. Let $w \neq 0$. Consider the Peirce decomposition relative to w

$$A = wAw + wA(1-w) + (1-w)Aw + (1-w)A(1-w).$$

If $a \in (1-w)Aw$, then $(w+a)^2 = w+a$. So that $(w+a)^n = (w+a)$, for all positive integer n . By our hypothesis, this implies that $w+a \in C = wAw$. But since $w \in wAw$, $a \in wAw$. Thus we have obtained that $a \in (1-w)Aw$ implies $a \in wAw$ too. This requires that $a = 0$. Hence $(1-w)Aw = 0$. Similarly $wA(1-w) = 0$. Thus w is then central.

The Peirce-decomposition then reduces to $A = wA + A(1-w)$. Set $B = A(1-w)$. If $x \in B$, then $x^n \in B \cap wA$ for some $n > 0$. Thus $x^n = 0$. B is then a nil ideal.

Next, A being a right (left) unitary ring, $x \in xA$ (or $x \in Ax$) for all x . In particular for every x in B , $x \in xA$ (or $x \in Ax$). So that $x \in x(Aw + B)$ or $x \in (Aw + B)x$, for

each $x \in B$. But since, for each $x \in B$, wx or xw is zero, it follows that $x \in xB$ (or $x \in Bx$) for all x in B .

Therefore, for every x in B , there exists a y in B such that $x = xy$ (or $x = yx$). This implies $x = xy^n$ (or $x = y^n x$), for every integer $n > 0$.

Since B is a nil ideal, it follows that $x = 0$ for all x in B . So that $B = 0$. Hence $A = Aw = wA$ and it is immediate that $A = Ae = eA$. This completes the proof.

2. Following exactly BROWN and MCCOY [1] for obtaining the maximal regular ideal of a ring we can show that the join of all right (left) unitary ideals is a right (left) unitary ideal and hence there exists a unique maximal right (left) unitary ideal $M = M(A)$ in A . The properties of $M(A)$: (i) $M(A/M(A)) = 0$ and (ii) If B is an ideal of A , then $M(B) \subseteq B \cap M(A)$, are also easy to verify. We shall prove the corresponding results for the maximal Π -unitary ideal in a ring.

We first prove the following

Lemma. *If to each element a of a ring A there corresponds a positive integer $n = n(a)$ and a central element $x = x(a)$ of A such that $a^n - a^n x$ is Π -unitary then a is Π -unitary.*

PROOF. By hypothesis there exists a positive integer m and a central element y of R such that

$$(a^n - a^n x)^m y = (a^n - a^n y)^m$$

so that

$$a^{mn}(1-x)^m y = a^{mn}(1-y)^m$$

where 1 is formal. But then

$$a^{mn}(1-x)^m(1-y) = 0.$$

So we get

$$a^{mn}(1-z) = 0,$$

where z is obviously in the center. Thus a is Π -unitary and this completes the proof.

We shall indicate by (a) the principal ideal in B generated by a .

We now prove the following theorem.

Theorem 2. *If M is the set of all elements a of R such that (a) is Π -unitary then M is an ideal in R .*

PROOF. If $z \in M$ and $t \in R$, then $zt \in M$, since $(zt) \subset (z)$. Similarly, $tz \in M$. If $z, w \in M$ and $a \in (z-w)$, then $a = u-v$ for some u in (z) and v in (w) . Since (z) is Π -unitary, $u^{n(u)} = u^{n(u)}r$ for some element r of R . Then

$$a^n - a^n r = (u-v)^n - (u-v)^n r = \text{terms each of which has } v \text{ as one of its factors.}$$

Since $v \in (w)$, we obtain that $a^n - a^n r \in (w)$ and is therefore Π -unitary. The above lemma now implies that a is Π -unitary and hence $z-w \in M$. This completes the proof.

Theorem 2.1. *If R is any ring, $M(R/M(R)) = 0$.*

PROOF. Let $\bar{a}$ denote the residue class modulo $M(R)$ which contains the element a of R . If $\bar{b} \in M(R/M(R))$ and $\bar{a} \in (\bar{b})$, then $a \in (b)$. Since $(\bar{b})$ is a Π -unitary ideal in $R/M(R)$, $\bar{a}$ is Π -unitary. For

$$\bar{a}^n = \bar{a}^n \bar{x}, \text{ then } a^n - a^n x \text{ is in } M(R).$$

Therefore, $a^n - a^n x$ is Π -unitary and hence a is Π -unitary. But a is an arbitrary element of (b) . It follows then (b) is a Π -unitary ideal.

Hence $b \in M(R)$ and therefore $\bar{b} = 0$, completing the proof.

The case $R = M(R)$. Let R be finite Π -unitary ring having $a_1, a_2, \dots, a_n$ as its elements. For each a_i we have a positive integer n_i and a central element b_i such that

$$a_i^{n_i} b_i = a_i^{n_i} \quad \text{i. e.} \quad a_i^{n_i} (1 - b_i) = 0,$$

where 1 is formal.

Let us define

$$c = 1 - (1 - b_1)(1 - b_2) \dots (1 - b_n).$$

Then

$$\begin{aligned} a_i^{n_i} c &= a_i^{n_i} - (1 - b_1) \dots (1 - b_n) a_i^{n_i} \\ &= a_i^{n_i} - (1 - b_1)(1 - b_2) \dots (1 - b_n)(1 - b_i) a_i^{n_i} \\ &= a_i^{n_i}, \text{ for } b_i \text{'s are all central and } a_i^{n_i} (1 - b_i) = 0. \end{aligned}$$

Evidently c which is a product of central elements is also central.

We have, thus, proved the following

Theorem 2.2. *A finite Π -unitary ring R has a central element c such that $a^{n(a)}c = a^{n(a)}$, for each a in R .*

Corollary. *A finite non-nil Π -unitary ring has a nil ideal modulo which it has a unity element.*

Finally we prove:

Theorem 3.1. *If for every x in a non-nil ring A there exists a positive integer $n = n(x)$ such that $x^n \in C$ and if every proper ideal is contained in a modular right (left) ideal then A has a unity element.*

PROOF. As in theorem 1.1 we have a nil ideal B in A such that

$$A = Aw + B.$$

The ideal Aw cannot be zero, for otherwise A is nil. If Aw were different from A , it can by hypothesis be imbedded in a modular right (left) ideal and therefore by Zorn Lemma it can be imbedded in a modular maximal right (left) ideal F . If R is the Jacobson radical then the nil ideal.

$$\begin{aligned} B &\subseteq R \\ &\subseteq F \end{aligned}$$

and thus we are led to a contradiction

$$A = Aw + B \subseteq F \subset A.$$

Hence $A = Aw = wA$ and it is immediate that $e = u$.

This completes the proof of the theorem.

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References

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