Rings of finite rank

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Let R be a ring and $L_r = L_r(R)$ be its lattice of right ideals. The *right rank* of R, $r(R)$, is defined to be max. card $S^\perp$, where $S^\perp$ is an independent subset of L_r. The *left rank* of R, $l(R)$, is defined similarly. If R is a zero ring, then $r(R) = l(R) = k$ where k is the usual rank of an Abelian group. Our remarks are restriced in this note to rings of finite right rank.

Associated with the lattice L_r of ring R is another lattice L_r' defined as follows. For $A, B \in L_r$, let $A \subset' B$ signify that B is an essential extension of A; that is, $A \subset B$ and $A \cap C \neq 0$ whenever $B \cap C \neq 0$, $C \in L_r$. Define the relation $\sim$ in L_r by: $A \sim B$ iff $A \cap B \subset' A$ and $A \cap B \subset' B$. It is readily shown that $\sim$ is an equivalence relation. Let $L_r' = L_r/\sim$ and φ be the natural mapping of L_r onto L_r'. If the partial ordering $\leqq$ is defined in L_r' by: $\varphi A \leqq \varphi B$ iff $A \cap B \subset' A$, then UTUMI showed in [1] that L_r' is a complemented modular lattice. Furthermore, he showed that φ is a meet homomorphism of L_r onto L_r'. Evidently $\varphi 0 = \{0\}$ and $\varphi R = \{A \in L_r | A \subset' R\}$. It may be shown that $\varphi(A \cup B) = \varphi A \cup \varphi B$ if $A \cap B = 0$. From these remarks, it is clear that $S^\perp$ in L_r iff $(\varphi S)^\perp$ in L_r'. Consequently, $r(R)$ is simply the dimension of lattice L_r'. Hence, $r(R) = \text{card } S^\perp$ where $S^\perp$ is any maximal independent subset of L_r.

Each right ideal A of ring R has a rank defined by $r(A) = \dim(\varphi A)$ in L_r'. The *right rank* of an element x of R, $r(x)$, is defined to be $r(A)$ where A is the right ideal of R generated by x. It is evident that

$$r(xy) \leqq r(x), \quad r(x+y) \leqq r(x) + r(y)$$

for all $x, y \in R$.

It does not seem possible to say much about the rank of the elements of a general ring of finite rank. However, if we assume that R has zero singular ideal, $R_r^\triangle = 0$, then some of the familiar properties of rank hold. We recall that $R_r^\triangle = \{x \in R | x^r \subset' R\}$, where x^r is the right annihilator of x in R. Let us call R an *F_r-ring* if R has finite right rank and $R_r^\triangle = 0$. For an F_r-ring R, it is easily shown that $r(x) = r(xC)$ for every $x \in R$ and every $C \in L_r$ such that $C \subset' R$.

If R is an F_r-ring, then each $A \in L_r$ has a unique maximal essential extension $A^* \in L_r$, called the closure of A. The set L_r^* of all closed right ideals of R is a lattice, which is easily shown to be isomorphic to L_r'. In fact, $\varphi A = \{B \in L_r | B \subset' A^*\}$ for each $A \in L_r$. Thus, $r(x) = r(xR) = \dim(xR)$ in L_r^*. Since $x^r \in L_r^*$ for each $x \in R$, and a maximal complement of each $A \in L_r$ is in L_r^*, it is evident that $r(R) = 1$ iff R is a right Ore domain.

Theorem 1. *If R is an F_r-ring, then $r(x) = r(R) - r(x^r)$ for every $x \in R$.*

PROOF. Let A be a complement of x^r in L_r^* and $\{A_1, \ldots, A_k\}$ be an atomic basis for A. Since $\{xA_1, \ldots, xA_k\}^{\perp}$, evidently $r(x) \geqq k = r(R) - r(x^r)$. If $\{xB_1, \ldots, xB_n\}^{\perp}$, where each B_i is an atom of L_r^*, and if $B = B_1 + \ldots + B_n$, then $B \cap x^r = 0$. For if $b = b_1 + \ldots + b_n \in x^r$, $b_i \in B_i$, then $xb = \sum xb_i = 0$ and $xb_i = 0$ for each i. However, $x^r \cap B_i = 0$ by assumption, and therefore $b_i = 0$ for each i and $b = 0$. Thus, B is contained in a maximal complement of x^r and $n \leqq k$. Consequently, $r(x) \leqq k$ and the theorem is proved.

Since $(xy)^r \supset y^r$, evidently

$$r(xy) \leqq r(y)$$

for all x, y in an F_r-ring by Theorem 1.

Theorem 2. *If R is an F_r-ring and $x, y \in R$ are such that $xR \cap yR = 0$, then $r(x+y) \geqq r(x)$. If, furthermore, $x^r + y^r \subset' R$ then $r(x+y) = r(x) + r(y)$.*

PROOF. If $r(x) = k$ and $\{xA_1, \ldots, xA_k\}^{\perp}$, A_i atoms of L_r^*, then $\{(x+y)A_1, \ldots$ $\ldots, (x+y)A_k\}^{\perp}$. For if $\sum (x+y)a_i = 0$, $a_i \in A_i$, then $\sum xa_i = -\sum ya_i = 0$ and $xa_i = 0$ for each i. Hence, $a_i = 0$ for each i. Therefore, $r(x+y) \geqq r(x)$.

To prove the second part, let B and C be relative complements of $x^r \cap y^r$ in x^r and y^r, respectively, and let $\{B_1, \ldots, B_k\}$, $\{C_1, \ldots, C_m\}$, and $\{D_1, \ldots, D_n\}$ be atomic bases of B, C, and $x^r \cap y^r$, respectively. If $B' = B_1 + \ldots + B_k$, $C' = C_1 + \ldots$ $\ldots + C_m$, and $D' = D_1 + \ldots + D_n$, then $B' + C' + D' \subset' R$ and $r(x+y) =$ $= r[(x+y)(B' + C' + D')] = r[(x+y)(B' + C')]$. Since $(x+y)B' = yB'$ and $(x+y)C' = xC'$, evidently $(x+y)B' \cap (x+y)C' = 0$. Hence, $r(x+y) = k + m =$ $= (k+m+n-k-n) + (k+m+n-m-n) = r(x) + r(y)$ in view of Theorem 1. This proves Theorem 2.

If R is an F_r-ring and $U = \{u \in R \mid uR \subset' R\}$, then U is a multiplicative semigroup by [2; 3.2]. Also, by [2; 3. 3], $u^r = u^l = 0$ for every $u \in U$. Since $(ux)^r = x^r$ for all $u \in U$ and $x \in R$, evidently $r(ux) = r(x)$. If $r(x) = k$ and $\{xA_1, \ldots, xA_k\}^{\perp}$, A_i atoms of L_r^*, then for each $u \in U$ we can select $B_i \in L_r$ such that $uR \cap A_i = uB_i$ for each i. Since $(uB_i)^* = A_i$, evidently $xuB_i \neq 0$ for each i. Consequently, $r(xu) = k$. We have proved that

$$r(xu) = r(ux) = r(x)$$

for all $x \in R$ and $u \in U$.

Let us call an F_r-ring R an *I_r-ring* if every $A \in L_r^*$ contains an element a such that $aR \subset' A$. Thus, an I_r-ring is a generalization of a principal right ideal ring. If R is an I_r-ring then for each $A \in L_r^*$, $r(A) = r(a)$ for some $a \in A$. In particular, $U \neq \Phi$ for an I_r-ring. If $\{R_1, \ldots, R_n\}$ is a set of I_r-rings, then their direct product $R_1 \times \ldots \times R_n$ is easily seen to be an I_r-ring. Also, if Q is a (right) quotient ring of an I_r-ring R (so that $qR \cap R \neq 0$ for each nonzero $q \in Q$), then Q is an I_r-ring (see [3]).

An F_r-ring is called (right) irreducible [3] iff $\{0, R\}$ is the center of lattice L_r^*. If R is not irreducible, then the center C_r^* of L_r^* is a Boolean algebra and each atom of C_r^* is an irreducible ring. If $\{S_1, \ldots, S_n\}$ is the set of atoms of C_r^* and $S = S_1 + \ldots$ $\ldots + S_n$ (a direct sum), then R is a quotient ring of S. Evidently R is an I_r-ring iff every S_i is an I_r-ring. Thus, the problem of describing I_r-rings reduces to that of describing irreducible I_r-rings.

In a forthcoming paper [4], an F_r-ring R is called *right potent* iff $A^2 \neq 0$ for every atom $A \in L_r^*$. This is equivalent to saying that no nonzero ideal of L_r^* is nilpotent. Let us call a right potent, irreducible F_r-ring a *P_r-ring*.

Theorem 3. *If R is a P_r-ring, then each $A \in L_r^*$ contains an element a such that $A \dot{\cup} a^r = R$.*

PROOF. The notation $\dot{\cup}$ is used for a direct union in lattice L_r^*. Let $r(R) = n$. If $n = 1$, then R is a right Ore domain and the theorem is trivially true. So let us assume that $n > 1$. If $A \in L_r^*$ is an atom, so that $r(A) = 1$, then $A \cap A^r = 0$ and $A \cap a^r = 0$ for some nonzero $a \in A$. Since a^r is a maximal element $(\neq R)$ of L_r^*, evidently $A \cup a^r = R$ in this case.

Assume that the integer $k > 1$ is chosen so that the theorem is true for every element of L_r^* of rank $< k$, and let $A \in L_r^*$, $r(A) = k$. Select $B \in L_r^*$ such that $B \subset A$ and $r(B) = k-1$. By assumption, there exists some $b \in B$ such that $B \dot{\cup} b^r = R$. Since $r(b^r) = n-k+1$ and $B \cap b^r = 0$, evidently $b^r \cap A = C$ where C is an atom of L_r^*. Let us select a nonzero $c \in C$ such that $c^r \cap C = 0$, and then let $a = b + c$. Clearly $bR \cap cR = 0$ and also $b^r + c^r \subset' R$. Hence, $r(a) = k$ by Theorem 2. If $x + y \in a^r \cap (B + C)$, with $x \in B$ and $y \in C$, then $(b+c)(x+y) = 0$ and $b(x+y) = -c(x+y) = 0$. Therefore, $b(x+y) = 0$, $bx = 0$, and $x = 0$; and $cy = 0$ and $y = 0$. We conclude that $a^r \cap (B+C) = 0$ and consequently that $a^r \cap A = 0$. Hence, $a^r \dot{\cup} A = R$. The theorem now follows by mathematical induction.

Corollary. *If R is a P_r-ring, then R is an I_r-ring. In fact, for each $A \in L_r^*$ there exist $a \in A$ and $b \in a^r$ such that $a + b \in U$.*

PROOF. Select $a \in A$ so that $A \dot{\cup} a^r = R$ and $b \in a^r$ so that $a^r \dot{\cup} b^r = R$. Then $(a+b)^r = a^r \cap b^r = 0$ and $a + b \in U$ by [2; 3. 4].

An I_r-ring need not be a P_r-ring. Consider, for example, the ring R of all matrices of the form

$$\begin{pmatrix} a+c & 0 \\ b & c \end{pmatrix}$$

where $c \in F$, a field, and $a, b \in xF[x]$. Since the 2×2 matrix ring $(F(x))_2$ is a quotient ring of R, we must have $R_r^\Delta = 0$ and $r(R) = 2$. Every atom of L_r^* contains elements of rank 1 and the only element of L_r^* of rank 2, R, has a unity and hence contains an element of rank 2. Thus, R is an I_r-ring. However, R is not a P_r-ring since $A = e_{21}R$ is an atom of L_r^* such that $A^2 = 0$.

The ring R of $n \times n$ triangular matrices over a field is an example of a P_r-ring (and P_l-ring) which is not a principal right ideal ring. However, R is a Baer ring; i. e., every annihilating right ideal of R is generated by an idempotent. In this example, L_r^* is precisely the set of annihilating right ideals.

Let R be an F_r- and F_l-ring, R_r^0 be the union in L_r of the atoms of L_r^*, and R_l^0 be the corresponding union in L_l^*. The ring R is called *stable* in [5] if $(R_r^0)^r = (R_l^0)^l = 0$. It is proved in [5; 3. 1] that if R is stable then the lattices L_r^* and L_l^* are dual isomorphic under the correspondence $A \to A^l$, $A \in L_r^*$. Hence, $r(R) = l(R)$ if R is stable.

Theorem 4. *If R is a stable ring, then $r(x) = l(x)$ for every $x \in R$.*

PROOF. By Theorem 1, $r(x) = r(R) - r(x^r)$. Since $x^{rl} = (Rx)^*$ and $r(x^r) = r(R) - l[(Rx)^*]$ by the dual isomorphism between L_r^* and L_l^*, we have $r(x) = l[(Rx)^*] = l(x)$ as desired.

References

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