

Cohesive groups and p -adic integers

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§ 1. Introduction

The construction of indecomposable torsionfree Abelian groups of rank two, due to PONTRIAGIN ([7]; see also [6] Theorem 19 and [2] p. 151) was generalized by CORNER [8] p. 696 to obtain groups of rank $c = 2^{\aleph_0}$. In § 3 present a construction which slightly generalizes Corner's, produces indecomposable torsionfree groups of rank up to c , and uses an arbitrary set of primes. If this set is the set of all primes, then the groups obtained have a property which was first noticed by SASIADA and JAREK for pure subgroups of $I(p)$, and which we call *cohesiveness*. This very strong property is examined in § 2, where we show that every reduced cohesive group is strongly and absolutely indecomposable and give in Theorem 1 a simple computational criterion for cohesiveness. A corollary is that the quasi-isomorphism class of a cohesive group coincides with its isomorphism class.

Notation. For most terms see FUCHS [2] or KAPLANSKY [6]. Z is the additive group of the rational integers, $I(p)$ the additive group of all p -adic integers. For a group or set S , $|S|$ is the power or cardinal number of S . A subgroup K of a group G is *p -pure in G* if and only if $K \cap p^n G = p^n K$ for $n = 0, 1, 2, \dots$; K is *pure in G* if and only if it is p -pure for every prime p . In case G is torsionfree and K is a subset of G , then the *pure (p -pure) closure of K in G* is the intersection of all pure (p -pure) subgroups containing K . A *rigid system* is a set S of torsionfree groups such that for all A and B in S , $\text{Hom}(A, B)$ is zero if $A \neq B$, and $\text{Hom}(A, A)$ has rank one. The height of x at the prime p is denoted by $H_p(x)$. See FUCHS [2], § 42, for discussion of heights and types in torsionfree groups.

§ 2. Cohesive groups

Definition. The torsionfree group G is *cohesive* if and only if for every non-zero pure subgroup S of G , G/S is divisible.

First we give some useful lemmas and examples.

C1. Every divisible torsionfree group is cohesive.

C2. A cohesive group is either divisible or reduced and any reduced cohesive group is indecomposable.

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C3. A torsionfree group G is cohesive if and only if every nonzero map of G into a reduced torsionfree group is a monomorphism (zero kernel).

C4. If $G \subset H \subset G^* \subset K$, if G is cohesive, if K is torsionfree, and if G^* is the pure closure of G in K , then H is also cohesive. From this it follows readily that cohesiveness is a quasiisomorphism invariant and that every reduced cohesive group is strongly indecomposable.

C5. Every pure subgroup of a cohesive group is cohesive; thus a reduced cohesive group is absolutely indecomposable, i. e., every pure subgroup is indecomposable.

C6. $I(p)$ is cohesive. For if S is a nonzero pure subgroup of $I(p)$, then it is immediately clear that S contains a unit; this implies $I(p)/S$ is divisible.

C7. The union of a chain of cohesive subgroups of a torsionfree group is cohesive.

C8. In a cohesive group all nonzero elements have types with the same set of infinity places. For suppose that G is cohesive and that a and c are nonzero members with $H_p(a) = \infty$, $H_p(c) = 0$. Let A be the pure subgroup generated by a . Then G/A is divisible, so p divides $c + A$ in G/A . This means that there is an x with $px = c + a'$, a' in A . Since every element in A has infinite height at p , p divides a' and hence p divides c . This contradicts $H_p(c) = 0$. It follows that de Groot's absolutely indecomposable example G [4], § 7, is not cohesive; for in G there is a set $a_1, a_2, \dots$ of elements with a_i having infinite height only at the prime p_i , where $a_i \rightarrow p_i$ is a one to one correspondence.

C9. If S is a rigid system, none of whose groups is divisible by p , then the subgroup $G(S, a)$ of the direct sum $T = \sum_{A \in S} A$ generated by pT and all members of the form $\sum_{i=1}^n a(A_i)$ where a is a fixed function on S with $a(A)$ a member of A that is not divisible by p , and $A_i \in S$, is indecomposable (see HULANICKI [5] and FUCHS [3]) but not cohesive. For the projection of $G(S, a)$ onto a fixed A' in S induces a nonzero map of G into A' that is not a monomorphism (unless S is a singleton set). But A' is a reduced torsionfree group, so by C3 above, $G(S, a)$ is not cohesive.

We shall need the following three trivial lemmas on p -adic integers.

P1. If S is torsionfree and $pS \neq S$ (where p is a prime), then there exist nonzero maps of S into $I(p)$. For let $r = \text{rank of } S/pS$. Then

$$\begin{aligned} \text{Hom}(S, I(p)) &= \text{Hom}(S, \text{Hom}(Z(p^\infty), Z(p^\infty))) \\ &\approx \text{Hom}(S \otimes Z(p^\infty), Z(p^\infty)) \approx \text{Hom}\left(\sum_r Z(p^\infty), Z(p^\infty)\right) \\ &\approx \prod_r (\text{Hom}(Z(p^\infty), Z(p^\infty))) = \prod_r I(p). \end{aligned}$$

This is zero if and only if $pS = S$.

P2. Every endomorphism of $I(p)$ is a multiplication by a fixed member of $I(p)$. See FUCHS [2], p. 212.

P3. If S is p -pure in $I(p)$, then every map of S into $I(p)$ is the restriction of an endomorphism of $I(p)$ (ARMSTRONG [1]). Fuchs' argument cited above can be generalized to prove this, as follows. Since S (assumed nonzero) is p -pure it contains a unit u . If U is the subgroup generated by u then S/U is divisible by p , while $I(p)$

has no elements of infinite height at p . It follows readily that every map of S into $I(p)$ is determined by its value at u and the assertion is proved.

Now we are ready to prove a computational criterion.

Theorem 1. *Let S be a torsionfree abelian group. The following are equivalent:*

(a) S is cohesive (b) for every prime p , either S/pS is zero or S is isomorphic with a p -pure subgroup of $I(p)$; (c) for every prime p , either S/pS is zero, or else S has no elements of infinite height at p and S/pS is a cyclic group of order p ; (d) for every prime p , every nonzero map of S into $I(p)$ is a monomorphism.

PROOF. (i). If (a) then (b). Let S be cohesive and suppose $pS \neq S$. By P2, there is a nonzero map $\varphi: S \rightarrow I(p)$. By C3, φ is a monomorphism. Let the image of φ be written as $p^k S'$, where S' contains a unit $u = u_0 + u_1 p + \dots$, with $u_0 \neq 0$. Then S' is isomorphic with S . Let px belong to S' , with x in $I(p)$, $x = x_0 p^t + x_1 p^{t+1} + \dots$, and $x_0 \neq 0$. Let X be the pure subgroup of S' generated by px . Then S'/X is divisible, so the coset $u + X$ is divisible by p . Then there is a rational r with rpX in S' and $u + rpX$ divisible by p in S' . Divisibility in $I(p)$ implies $r = k/(np^{t+1})$, with k and n integers prime to p . Hence $kx = np^t(rpX)$ belongs to S' . Take integers a and b with $ak + bp = 1$, so that $x = a(kx) + b(px)$ belongs to S' . Thus S' is p -pure in $I(p)$ and the first implication is proved.

(ii). If (b) then (c). Let S satisfy condition (b) and suppose $pS \neq S$. From (b) we may assume that S is a p -pure subgroup of $I(p)$. Then S has no elements of infinite height at p since $I(p)$ has none. If $u = u_0 + u_1 p + \dots$ is any unit in S then $u + pS$ generates S/pS . For given x in S choose m with mu_0 congruent to $x_0 \pmod{p}$ (m an integer). Then p divides $x - mu$. This shows (c).

(iii). If (c) then (d). Let S satisfy condition (c) and let p be a prime. If $pS = S$ then S has no nonzero maps into $I(p)$ so we suppose that S/pS is cyclic, generated by $a + pS$, and that S has no elements of infinite height at p . For every x in S a sequence x_i of members of S and a sequence n_i of integers is uniquely determined by the conditions: $x = n_0 a + px_1$, $0 \leq n_0 < p$, $x_i = n_i a + px_{i+1}$, $0 \leq n_i < p$, $i = 1, 2, \dots$. Then for every k , $x = \left(\sum_{i=0}^k n_i p^i \right) a + p^{k+1} x_{k+1}$ and the correspondence $x \rightarrow \sum_{i=0}^{\infty} n_i p^i$ defines a homomorphism φ from S into $I(p)$. The image of φ is p -pure in $I(p)$ since if $py = x$ then $x = 0 \cdot a + px_1$ and $x_1 \varphi = y$, and the kernel of φ is the set of all elements of infinite height at p , hence zero. In view of P2 and P3, the proof of (iii) is complete.

(iv). If (d) then (a). Suppose that S satisfies (d), that $\varphi: S \rightarrow T$ is a nonzero epimorphism, and that T is a reduced torsionfree group. Since T is reduced there is a prime p with $pT \neq T$, and so there is a nonzero homomorphism $\psi: T \rightarrow I(p)$. Then $\varphi\psi: S \rightarrow I(p)$ is not zero. By (d), $\varphi\psi$ is a monomorphism; therefore φ is also a monomorphism. This proves that S is cohesive, i. e., condition (a). The theorem is proved.

G is quasi-isomorphic with H if G is isomorphic with a subgroup H' of H which contains a nonzero multiple mH of H .

Corollary. *Let G be quasi-isomorphic with the cohesive group H . Then G is isomorphic with H .*

PROOF. Suppose $mH \subset H' \subset H$ with $m \neq 0$. If m is a prime then it follows immediately from Theorem 1 (c) that H' is either H or mH . An induction shows that in any case $H' = kH$ where k is a factor of m , and hence that H' is isomorphic with H .

Note that the largest possible power of a reduced cohesive group is $c = 2^{\aleph_0}$.

For an example of a p -pure subgroup of $I(p)$ that is not cohesive, let x and y be independent in $I(p)$, let q be a prime different from p , and let G be the p -pure closure in $I(p)$ of the group generated by y and all $q^{-n}x$, $n = 0, 1, 2, \dots$. Then G is not cohesive since x has infinite height at q while y has finite height at q .

Theorem 2. *Let A be a torsionfree reduced cohesive group of rank at least two. Then $A \otimes A$ is not cohesive.*

PROOF. Let a and b be independent in A , and let F be the free subgroup of A generated by a and b . Then the natural map of $F \otimes F$ into $A \otimes A$ is a monomorphism. In $F \otimes F$ the elements $a \otimes a, a \otimes b, b \otimes a$ and $b \otimes b$ are independent generators, so $a \otimes b - b \otimes a$ is not zero in $A \otimes A$. Now choose a prime p so that $pA \neq A$ and choose g in A with $g + pA$ a generator of A/pA . This is possible by Theorem 1 (note that A is assumed reduced). Let k be a positive integer. By the construction in Theorem 1, part (iii), we can write $a = mg + p^k a'$, $b = ng + p^k b'$. Then clearly $a \otimes b - b \otimes a$ is divisible by p^k . This proves that $H_p(a \otimes b - b \otimes a) = \infty$. Next identify A with a subgroup of $I(p)$ and let φ be the homomorphism of $A \otimes A$ into $I(p)$ defined by the formula: $(x \otimes y)\varphi = xy$. Then φ is not zero and therefore $p(A \otimes A) \neq A \otimes A$, since $I(p)$ has no elements of infinite height at p . Thus it has been shown that $p(A \otimes A) \neq A \otimes A$, but not all (nonzero) elements of $A \otimes A$ have finite height at p . By Theorem 1, $A \otimes A$ is not cohesive.

Theorem 3. *If A and B are cohesive then $\text{Hom}(A, B)$ is cohesive.*

PROOF. Set $H = \text{Hom}(A, B)$. If either A or B is divisible by a prime p then so is H . Suppose H not divisible by p ; then neither A nor B is divisible by p , so each of A and B has, by cohesiveness, no elements of infinite height at p . If φ belongs to H and has infinite height at p in H , then the image of φ in B is divisible by p ; hence φ is zero. Thus H contains no elements of infinite height at p . Let φ have height zero at p . Then $\text{Im } \varphi$ is not contained in pB ; choose a in A so that $a\varphi$ has height zero at p . Then a likewise has height zero at p . Now we show that $\varphi + pH$ generates H/pH . Let ψ belong to H . Since p does not divide $a\varphi$, $a\varphi + pB$ generates B/pB , whence $a\psi - ma\varphi = pb'$ ($b' \in B$) for some integer m . For arbitrary x in A , write $x = na + px_1$. Then

$$x(\psi - m\varphi) = p(nb' + x_1\psi - mx_1\varphi).$$

Hence $\psi - m\varphi$ is divisible by p in H . By Theorem 1, H is cohesive.

§ 3. Constructions

We construct a group $G(S)$ where S is a set of functions denoted by π , with values π_p , with the following properties:

S1. $1 \leq |S| \leq c$.

S2. Each function has the same domain, denoted by $D(S)$, a nonempty set of primes.

S3. For each prime p in $D(S)$, the set S_p , where $S_p = \{\pi_p; \pi \in S\}$, is an algebraically independent subset (over the rationals) of $I(p)$; and if $\pi_p = \pi'_p$, then $\pi = \pi'$.

To construct such a set S , let L be a nonempty set of primes and for each p in L let T_p be a well-ordered algebraically independent set of p -adic integers with order type Γ , where Γ is the least ordinal of power c . For each ordinal α with $0 \leq \alpha < \Gamma$ set $\pi_\alpha(p)$ equal to the α^{th} member of T_p . Take an ordinal $\beta \leq \Gamma$ and set $S = \{\pi_\alpha; \alpha < \beta\}$. Then S satisfies all requirements, $S_p = T_p$, $D(S) = L$, and $|S| = |\beta|$.

Let M be a linearly independent set of reals strictly between 0 and 1, with y in M and $|M| = |S| + 1$. Let x be a one to one function mapping S onto $M - \{y\}$. For each π in S and p in $D(S)$ write the standard p -adic power series:

$$\pi_p = \pi_{p0} + \pi_{p1}p + \dots,$$

with $0 \leq \pi_{pi} < p$, and set $\pi_p^0 = 0$, $\pi_p^n = \sum_{i=0}^{n-1} \pi_{pi} p^i$ for $n > 0$. Then $G(S)$ is the group of real numbers generated by y and all $x(\pi)_p^n$, for $n = 0, 1, 2, \dots$ and p in $D(S)$, where

$$(1) \quad x(\pi)_p^n = p^{-n}(x(\pi) + \pi_p^n y).$$

Note that $\pi_{pk} = p^{-k}(\pi_p^{k+1} - \pi_p^k)$, and that, for $k = 0, 1, 2, \dots$,

$$(2) \quad px(\pi)_p^{k+1} = x(\pi)_p^k + \pi_{pk} y.$$

Members of $G(S)$ are simply those reals that are equal to a formula

$$(a) \quad a'y + \sum_{\pi} \sum_p \sum_{n=0}^{k(\pi, p, a)} a_{\pi pn} x(\pi)_p^n,$$

where the coefficients a' and $a_{\pi pn}$ are integers and almost all of them are zero; in the sums, π runs over S and p over $D(S)$. If $D(S)$ is a singleton, say $D(S) = \{p\}$, then $G(S)$ is the Fuchs example of type zero, Pontriagin's example if $p = 2$. Every element b of $G(S)$ can be written (uniquely) in the form (almost all coefficients are zero)

$$(3) \quad b = b'y + \sum_{\pi} b_{\pi} x(\pi)$$

where b' and b_{π} are rationals.

GS1. Let the coefficients of b in (3) be integers. If p belongs to $D(S)$ then p^t divides b in $G(S)$ if and only if p^t divides $b' - \sum_{\pi} b_{\pi} \pi_p^t$ in $\mathbb{Z}$. If p does not belong to $D(S)$, then p^t divides b if and only if p^t divides every coefficient.

PROOF. First let the coefficients of b be rationals, and suppose b belongs to $G(S)$. Then b is equal to a formula (a). By independence of the set M we get

$$b' = a' + \sum_{\pi} \sum_p \sum_{n=0}^{k(\pi, p, a)} a_{\pi pn} p^{-n} \pi_p^n,$$

$$b_{\pi} = \sum_p \sum_{n=0}^{k(\pi, p, a)} a_{\pi pn} p^{-n} \quad (\pi \in S).$$

Let, for each prime q , W_q be the (logarithmic) q -adic valuation. (If r is a rational, write $r = q^t r'$ where r' is a rational with numerator and denominator prime to q , and set $W_q(r) = t$; then $W_q(r+s) \geq \min \{W_q(r), W_q(s)\}$, $W_q(rs) = W_q(r) + W_q(s)$.) If q is a prime not in $D(S)$, then both right sides in equations above have nonnegative q -adic valuation, and so $W_q(b') \geq 0$, $W_q(b_\pi) \geq 0$ for all π . This proves the second part. Now suppose that q belongs to $D(S)$, let π be a member of S , set $m = k(\pi, q, a)$ and suppose that $m > -W_q(b_\pi)$, $m > 0$. Equating the q -adic values of both sides in the last equation above shows that $a_{\pi q m}$ is divisible by q , say $a_{\pi q m} = qd$. If this is substituted in formula (a) and equation (2) is used, a new formula (c) is obtained with $k(\pi, q, c) < k(\pi, q, a)$ and $k(\pi', p, c) = k(\pi', p, a)$ if $(\pi', p) \neq (\pi, q)$. By induction, there exists a formula (a) for b with

$$k(\pi, p, a) \geq \max \{0, -W_p(b_\pi)\}$$

for all π and all p in $D(S)$.

Now suppose that b has integral coefficients, and that p^t divides b in $G(S)$. Since $x(\pi)_q^0 = x(\pi) = x(\pi)_p^0$, the result just obtained guarantees that for some integers a' , $a_{\pi p n}$,

$$b = p^t \left(a'y + \sum_{n=0}^t a_{\pi p n} x(\pi)_p^n \right),$$

so that the equations expressing independence have the simpler form

$$b' = p^t \left(a' + \sum_{\pi} \sum_{n=0}^t \pi_p^n p^{-n} a_{\pi p n} \right), \quad b_\pi = \sum_{n=0}^t p^{t-n} a_{\pi p n}.$$

Multiply the last equation by π_p^t , sum on π , and subtract from the next to last equation. This gives

$$b' - \sum_{\pi} b_\pi \pi_p^t = p^t \left(a' + \sum_{\pi} \sum_{n=0}^t a_{\pi p n} p^{-n} (\pi_p^n - \pi_p^t) \right),$$

and p^n divides $\pi_p^n - \pi_p^t$ for all $n \leq t$. Hence the "only if" part of the first assertion is proved. If, conversely, $b' - \sum_{\pi} b_\pi \pi_p^t = p^t c$, with integral c , then a straightforward computation shows that

$$b = p^t (cy + \sum_{\pi} b_\pi x(\pi)_p^t),$$

so p^t divides b in $G(S)$.

GS2. Every nonzero element of $G(S)$ has finite height at every prime.

PROOF. It is sufficient to prove that if b , in equation (3), has integral coefficients and infinite height at p , then $b=0$. This is immediate if p is not in $D(S)$. If p belongs to $D(S)$, then, by GS1, p^t divides $b' - \sum_{\pi} b_\pi \pi_p^t$ for every t . In the p -adic completion of the rationals, therefore,

$$0 = \lim (b' - \sum_{\pi} b_\pi \pi_p^t) = b' - \sum_{\pi} b_\pi \pi_p.$$

This equation and the algebraic independence of the set S_q (recall that if $\pi_p = \pi'_p$ then $\pi = \pi'$) imply $0 = b' = b_\pi$ for all π , whence $b=0$.

GS3. If p belongs to $D(S)$ then $G(S)$ is isomorphic with a p -pure subgroup of $I(p)$ lying between the subgroup generated by 1 and S_p and the pure closure (in $I(p)$) of the latter subgroup. Thus $G(S)$ is always indecomposable.

PROOF. First we show that the coset of y generates $G(S)/pG(S)$, by showing that every generator of $G(S)$ is congruent to a natural multiple of $y \bmod pG(S)$, where p is any member of $D(S)$. Equation (2) shows this for generators of the form $x(\pi)_p^n$. Let q be a member of $D(S)$ different from p . In equation (2) replace p by q and subtract the result from equation (2). There results

$$q^n x(\pi)_q^n = (\pi_q^n - \pi_p^n)y + p^n x(\pi)_q^n;$$

since q is prime to p , $x(\pi)_q^n$ has the required property (for $n > 0$; $x(\pi)_q^0 = x(\pi)_p^0$). Next replace a by $-y$ in the proof of part (iii) of Theorem 1 to get a map φ ; since $G(S)$ has no elements of infinite height at p , by GS2, φ is a monomorphism whose image is p -pure in $I(p)$. The remaining assertion is proved by observing that $y\varphi = -1$ and that, from equation (2), $x(\pi)\varphi = \pi_p$ (see the construction of φ in part (iii) of Theorem 1).

GS4. $G(S)$ is cohesive if and only if $D(S)$ is the set of all primes.

PROOF. The "if" part follows from GS3 and Theorem 1 (b). Conversely, if p is not in $D(S)$ then, for arbitrary π in S , y and $x(\pi)$ are independent $\bmod pG(S)$, according to GS1. By Theorem 1 (c), $G(S)$ is not cohesive.

GS5. If $D(S)$ is finite then $G(S)$ is homogeneous of type zero. If for some π in S , $\pi_{p0} = \pi_p^1 = 1$ for all p in $D(S)$, and if $D(S)$ is infinite, then $G(S)$ is not homogeneous. If for all π in S and all p in $D(S)$, $\pi_{p0} = [(\log p)^{x(\pi)}]$, then $G(S)$ is homogeneous of type zero; $[t]$ is the greatest integer $\leq t$. (Another condition guaranteeing homogeneity is given by CORNER [8] p. 697.)

PROOF. GS1 implies the first assertion immediately, and with the hypothesis of the second, it implies that $y - x(\pi)$ is divisible by every prime in $D(S)$; since $D(S)$ is infinite, $y - x(\pi)$ has a nonzero type. But y always has type zero. To prove the third assertion we may suppose that $D(S)$ is infinite. Suppose $b = b'y + \sum_{\pi} b_{\pi}x(\pi) \neq 0$ has integral coefficients and nonzero type. Then b is divisible by infinitely many primes in $D(S)$ (only finitely many primes outside $D(S)$ can divide b). By GS1, there is an infinite set K of primes such that if p belongs to K , then p divides $b' - \sum_{\pi} b_{\pi}\pi_{p0}$. But for all large p in K ,

$$|b' - \sum_{\pi} b_{\pi}\pi_{p0}| = |b' - \sum_{\pi} b_{\pi}[(\log p)^{x(\pi)}]| \leq |b'| + \sum_{\pi} |b_{\pi}| \log p < p.$$

(Recall $0 < x(\pi) < 1$.) Hence there is an infinite subset K' of K such that if p belongs to K' then $b' - \sum_{\pi} b_{\pi}\pi_{p0} = 0$. There is a sequence p_n of primes in K' with $\lim p_n = \infty$, and, for every n ,

$$b' = \sum_{\pi} b_{\pi}[(\log p_n)^{x(\pi)}].$$

Let $m = \max \{x(\pi); b_{\pi} \neq 0\}$ (this set is not empty since y has type zero) and suppose $x(\pi') = m$. Then the right side is dominated, for large n , by the term $b_{\pi'}[(\log p_n)^m]$ (x is a one-to-one function) so that the right side tends to infinity contrary to the above equation. The contradiction shows that $G(S)$ is homogeneous of type zero.

Theorem 4. For every cardinal number m with $1 \leq m \leq c$ and every characteristic α , there exists a homogeneous cohesive group of type $[\alpha]$ and rank m and a nonhomogeneous cohesive group of rank m .

PROOF. All that is left to prove is the assertion about type $[\alpha]$. Let $G(S)$ be cohesive (by taking $D(S)$ to be the set of all primes) and homogeneous of type zero, rank m , and let H be the group of all multiples rt , where r belongs to the unique group K of rationals in which 1 appears and has height α , and t belongs to $G(S)$. Then H is still cohesive, rank m , and is homogeneous of type $[\alpha]$.

Theorem 5. Let W be an algebraically independent (over the rationals) subset of power c in $I(p)$ and let V be a family of power 2^c of pairwise incomparable (by set inclusion) subsets of W , each of power c . For each T in V let T' be a p -pure subgroup of $I(p)$ containing $\{1\} \cup T$ and contained in the pure closure of $\{1\} \cup T$. Then the set $V' = \{T'; T \in V\}$ is a rigid system.

PROOF. Let T and U be members of V and consider a nonzero map, necessarily a multiplication m_h , from T' to U' . Let t_1 be an arbitrary member of T and let t_2 be either 1 or a member of T different from t_1 . There exist u_i in U and rationals a, b, a_i, b_i with

$$t_1 h = a + \sum a_i u_i$$

$$t_2 h = b + \sum b_i u_i$$

$$t_1/t_2 = (a + \sum a_i u_i)/(b + \sum b_i u_i).$$

The last equation and the algebraic independence of W imply that t_1 belongs to U . Thus $T \subset U$ so, by hypothesis, $T = U$. The algebraic independence of T now shows that h is rational. This completes the proof.

From this and GS3 we get:

Corollary. A rigid system $\{G(S(T)); T \in V\}$ is obtained if V satisfies the hypothesis of Theorem 5 and, for each T in V , $(S(T))_p = T$.

References

- [1] J. ARMSTRONG, On p -pure subgroups of the p -adic integers, *Topics in Abelian Groups*, Chicago, 1963.
- [2] L. FUCHS, Abelian groups, *Budapest*, 1958.
- [3] L. FUCHS, On a directly indecomposable group of power greater than continuum, *Acta Math. Sci. Hung.* **8** (1957), 453–454.
- [4] J. DE GROOT, Indecomposable abelian groups, *Indagationes Math.* **19** (1957), 137–145.
- [5] A. HULANICKI, Note on a paper of de Groot, *Indagationes Math.* **20** (1958), 114.
- [6] I. KAPLANSKY, Infinite Abelian groups, *Ann Arbor*, 1954.
- [7] L. PONTRIAGIN, Theory of topological commutative groups, *Ann. of Math.* **35** (1934), 361–388.
- [8] A. L. S. CORNER, Every countable reduced torsionfree ring is an endomorphism ring, *Proc. London Math. Soc.* (3) **13** (1963), 687–710.

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