

On the summability of the Fourier series of L^2 integrable functions, II

By E. MAKAI (Budapest)

§ 1. Introduction

Let π_n and π'_n be the classes of the not identically vanishing n 'th order trigonometrical polynomials

$$(1.1) \quad f(x) = \frac{a_0}{2} + \sum_{v=1}^n (a_v \cos vx + b_v \sin vx)$$

and

$$(1.2) \quad f(x) = \sum_{v=1}^n (a_v \cos vx + b_v \sin vx)$$

respectively, $s_k(x) = s_k(x; f)$ the k 'th partial sum of $f(x)$ and $C_n^{(m)}$ the least positive quantity for which the inequality

$$(1.3) \quad \left| \frac{1}{m} \sum_{r=1}^m s_{k_r} \left(\frac{2\pi r}{m} \right) \right| \leq C_n^{(m)} \left\{ \frac{|a_0|^2}{2} + \sum_{v=1}^n (|a_v|^2 + |b_v|^2) \right\}^{\frac{1}{2}}$$

holds for each set of indices $k_1, k_2, \dots, k_m$ satisfying the inequalities $0 \leq k_r \leq n$ ($r=1, 2, \dots, m$) and for each function $f(x) \in \pi_n$.

The existence of two sequences of positive integers $\{m_s\}$ and $\{n_s\}$ ($s=1, 2, 3, \dots$) both tending to infinity, with the sequence $\{C_{n_s}^{(m_s)}\}$ remaining bounded, would imply that the Fourier series of each L^2 integrable function would converge almost everywhere. (Cfr. [4].) In [4] I have shown that if

$$(1.4) \quad E_l(x) = \cos x + \cos 2x + \dots + \cos lx, \quad E_0(x) = 0$$

then

$$(1.5) \quad C_n^{(m)} = \frac{1}{m} \max_{\substack{k_r=0, 1, \dots, n \\ r=1, 2, \dots, m}} \left\{ \frac{m^2}{2} + \sum_{p=1}^m \sum_{q=1}^m E_{\min(k_p, k_q)} \left(\frac{2\pi}{m} (p-q) \right) \right\}^{\frac{1}{2}}$$

and I have formulated the following

Conjecture (C): If $m|n$ (including the trivial case $n=0$) then

$$(1.6) \quad C_n^{(m)} = \sqrt{\frac{1}{2} + \frac{n}{m}}.$$

Moreover equality occurs in (1.3) if and only if

$$k_1 = k_2 = \dots = k_m = n$$

and $f(x)$ is a multiple of

$$(1.7) \quad \frac{1}{2} + \cos mx + \cos 2mx + \dots + \cos nx.$$

Note that the conjecture implies that $C_n^{(m)}$ is bounded. One of the purposes of this paper is to verify this conjecture for $m \leq 53$; for $m > 53$ its truth or falsity remains an open question.

The statement of the conjecture in the cases $m \leq 53$ will follow from the somewhat stronger

Statement (C') If $m|n$, $m \leq 53$ and $\lambda_n^{(m)}$ is the least positive quantity for which the inequality

$$(1.8) \quad \left| \sum_{r=1}^m s_{k_r} \left(\frac{2\pi}{m} r \right) \right| \leq \lambda_n^{(m)} \left\{ \sum_{v=1}^n (|a_v|^2 + |b_v|^2) \right\}^{\frac{1}{2}}$$

holds with each set of indices $k_1, \dots, k_m$ satisfying the inequalities $0 \leq k_r \leq n$ ($r = 1, 2, \dots, m$) and for each function $f(x) \in \pi_n'$ then

$$(1.9) \quad \lambda_n^{(m)} = \sqrt{nm}.$$

Moreover equality occurs in (1.8) if and only if

$$k_1 = k_2 = \dots = k_m = n$$

and $f(x)$ is a multiple of

$$(1.10) \quad E_{n/m}(mx) = \cos mx + \cos 2mx + \dots + \cos nx.$$

In the meanwhile we shall have occasion to solve the following

Problem. If π_n'' is the class of real trigonometrical polynomials of the form (1.2) normed by the condition

$$\sum_{v=1}^n (a_v^2 + b_v^2) = 1$$

and $1 \leq r_1 < r_2 < \dots < r_\mu \leq n \leq 38$ (r_κ integer, $\kappa = 1, 2, \dots, \mu$) to find the least quantity $\lambda_n'(r_1, r_2, \dots, r_\mu)$ for which

$$(1.11) \quad \sum_{\kappa=1}^{\mu} \max_{k=0,1,\dots,n} s_k \left(\frac{2\pi}{n} r_\kappa \right) \leq \lambda_n'(r_1, r_2, \dots, r_\mu) \quad (f \in \pi_n'')$$

and to find the extremal functions for which (1.11) holds with sign of equality.

The solution is:

$$(1.12) \quad \lambda_n'(r_1, r_2, \dots, r_\mu) = \sqrt{n\mu}$$

irrespectively of the distribution of the integers r_κ ; they may be in a cluster or may be distributed more or less uniformly in the interval $(1, n)$.

This problem has one and only one extremal polynomial $\hat{f}(x)$ characterized by the following peculiar properties:

- (i) $\hat{f}\left(\frac{2\pi}{n}r\right) = \sqrt{\frac{n}{\mu}}$ if $r \in (r_1, \dots, r_\mu)$;
- (ii) $\hat{f}\left(\frac{2\pi}{n}r\right) = 0$ if $r \in (1, 2, \dots, n)$ and $r \notin (r_1, \dots, r_\mu)$;
- (iii) $\hat{f}\left(\frac{2\pi}{n}\left(r + \frac{1}{2}\right)\right) = -\sqrt{\frac{\mu}{n}}$ if $r \in (1, 2, \dots, n)$;
- (iv) its Fourier coefficient b_n vanishes.

So the extremal polynomial assumes only three different values in the points $\pi j/n$ ($j=1, 2, \dots, 2n$; $n \leq 38$).

With the method of this paper it may be possible to extend this result to some other n 's, too, though this method falls short in treating the case of unrestricted n 's [1]. If the Problem's solution would be of the form indicated above for each n , it would imply Statement (C') for any m and Conjecture (C). Indeed Statement (C'), in the case of unrestricted n 's would follow by Lemma 2 from the special case $\mu|n, r_\kappa = \kappa n/\mu$. However, there exist examples showing that the solution of the Problem in the case $n > 38$ is not always given by formula (1.12) nor do the extremal functions always possess properties (i)–(iv).

On the other hand this Problem is solved in a host of cases. For, in the case of a fixed n , there exists $2^n - 1$ possibilities of choosing μ numbers $r_1, r_2, \dots, r_\mu$ ($\mu=1, 2, \dots, n$) from the set $1, 2, \dots, n$ and so the number of special cases in which the Problem is solved is $2^{39} - 39$.*).

The most important special case is, however $\mu=n$. Then we have the exceedingly simple results

$$\lambda'_n(1, 2, \dots, n) = n$$

and $\hat{f}(x) = \cos nx$ ($n \leq 38$). If this solution of the particular case $\mu=n$ would hold without the restriction $n \leq 38$ or at least for an infinity of n 's, this would again imply that the Fourier series of any L^2 integrable function would converge almost everywhere.

Now if $r_1, r_2, \dots, r_n$ is any permutation of the numbers $1, 2, \dots, n$ ($n > 38$), little can be said about the sequence

$$\lambda'_n(r_1), \lambda'_n(r_1, r_2), \dots, \lambda'_n(r_1, r_2, \dots, r_{n-1}), \lambda'_n(r_1, r_2, \dots, r_n)$$

except that it consists of nondecreasing numbers. Indeed if $\hat{f}(x) \in \pi''_n$ is one of the extremals defined above, i. e.:

$$\lambda'_n(r_1, r_2, \dots, r_\mu) = \sum_{\kappa=1}^{\mu} \max_{k=0, \dots, n} s_k \left(\frac{2\pi}{n} r_\kappa; \hat{f} \right)$$

*) If one identifies those special cases which are trivially equivalent, this number must be reduced by a factor of about 1/75.

($\mu < n$, n unrestricted) then one has

$$\begin{aligned}\lambda'_n(r_1, \dots, r_\mu, r_{\mu+1}) &\equiv \sum_{\kappa=1}^{\mu+1} \max_{k=0, 1, \dots, n} s_k \left(\frac{2\pi}{n} r_\kappa; \hat{f} \right) \equiv \\ &\equiv \sum_{\kappa=1}^{\mu} \max_{k=0, \dots, n} s_k \left(\frac{2\pi}{n} r_\kappa; \hat{f} \right) = \lambda'_n(r_1, \dots, r_\mu)\end{aligned}$$

since $s_0(2\pi r_{\mu+1}/n, \hat{f}) = 0$.

It may be possible that this trivial inequality could be sharpened to

$$\lambda'_n(r_1, \dots, r_\mu) < \lambda'_n(r_1, \dots, r_{\mu+1}).$$

If this would be true for $\mu = n-1$ only, this would be decisive. For we shall prove the following

Theorem 1. *If for some n the inequality*

$$\lambda'_n(1, 2, \dots, n-1) < \lambda'_n(1, 2, \dots, n)$$

holds, then for this n one has $\lambda'_n(1, 2, \dots, n) = n$.

Again, one sees from the solution of the Problem that in the case $\mu = n \leq 38$, the extremal function $\hat{f}(x) = \cos nx$ displays the following three features: (a) it is positive on the places $x = 2\pi j/n$ ($j = 1, 2, \dots, n$), (b) it is an even function and (c) it is unique. In connection with this we have

Theorem 2. *If for some n each extremal function of (1.11) in the case $\mu = n$ has property (a) or (b), or there exists only one extremal function, then for this n the equality $\lambda'_n(1, 2, \dots, n) = n$ holds.*

§ 2. Definitions, notations and lemmas

The notation $\|f\|$ will be used for any polynomial of the form (1.1) to denote the quantity

$$\left\{ \frac{|a_0|^2}{2} + \sum_{v=1}^n (|a_v|^2 + |b_v|^2) \right\}^{\frac{1}{2}}.$$

The classes π_n , π'_n and π''_n of trigonometrical polynomials have been already defined in § 1.

For sake of simplicity the m -tuple of integers $k_1, k_2, \dots, k_m$ will be called the vector $\mathbf{k}$ and the set of all vectors admissible in (1.5) (i. e. the set of all vectors the coordinates of which are nonnegative integers not exceeding n) the set K . Then if

$$(2.1) \quad E(k_1, \dots, k_m) = E(\mathbf{k}) = \sum_{p=1}^m \sum_{q=1}^m E_{\min(k_p, k_q)} \left(\frac{2\pi}{m} (p-q) \right),$$

the conjectured equality (1.6) is by virtue of (1.5) equivalent to

$$(2.2) \quad \max_{\mathbf{k} \in K} E(\mathbf{k}) = mn \quad (m|n).$$

A vector $\mathbf{k}'$ will be called a *maximal vector* if

$$E(\mathbf{k}') = \max_{\mathbf{k} \in K} E(\mathbf{k})$$

and the set of maximal vectors for a given m and n will be denoted by K^* .

Finally it will be convenient to introduce the $m \times m$ matrix $\mathbf{E}(\mathbf{k}) = [e_{pq}]_{p,q=1}^m$ with the elements

$$(2.3) \quad e_{pq} = E_{\min(k_p, k_q)} \left(\frac{2\pi}{m} (p - q) \right).$$

The scalar quantity $E(\mathbf{k})$ is equal to the sum of the elements of the matrix $\mathbf{E}(\mathbf{k})$.

We shall need the following lemmas.

Lemma 1. *If for some m and n ($m|n$) the statement (C') holds, then for the same particular values of m and n the conjecture (C) is also true.*

Lemma 2. *Let $\mathbf{x}$ be a vector, the elements of which are the real numbers $x_1, x_2, \dots, x_m$, n a natural number (not necessarily a multiple of m), $\mathbf{k} = \{k_1, k_2, \dots, k_m\}$ a vector the elements of which are nonnegative integers not exceeding n , $\lambda'(\mathbf{x}, \mathbf{k})$ and $\lambda''(\mathbf{x}, \mathbf{k})$ the least quantities for which the inequalities*

$$(2.4) \quad \left| \sum_{r=1}^m s_{k_r}(x_r; f) \right| \leq \lambda'(\mathbf{x}, \mathbf{k}) \quad (f \in \pi'_n, \|f\| = 1)$$

and

$$(2.5) \quad \sum_{r=1}^m s_{k_r}(x_r; f) \leq \lambda''(\mathbf{x}, \mathbf{k}) \quad (f \in \pi''_n)$$

respectively, hold. Then

$$\lambda'(\mathbf{x}, \mathbf{k}) = \lambda''(\mathbf{x}, \mathbf{k}) = \left\{ \sum_{p=1}^m \sum_{q=1}^m E_{\min(k_p, k_q)} (x_p - x_q) \right\}^{\frac{1}{2}}.$$

If $\lambda'(\mathbf{x}, \mathbf{k}) \neq 0$, then there exists but one extremal function in π''_n for which equality holds in (2.5). This is the polynomial

$$(2.6) \quad f_{\mathbf{x}, \mathbf{k}}(x) = [\lambda'(\mathbf{x}, \mathbf{k})]^{-1} \sum_{r=1}^m E_{k_r}(x - x_r)$$

and any extremal function in (2.4) is of the form $e^{i\alpha} f_{\mathbf{x}, \mathbf{k}}(x)$ where α is real.

Lemma 3. *If $m|n$, $\mathbf{k}$ is an admissible vector,*

$$k_1 \equiv k_2 \equiv \dots \equiv k_m \equiv 0 \pmod{m}$$

and $\mathbf{k}$ is not equal to the vector

$$(2.7) \quad \mathbf{k}^* = \{n, n, \dots, n\}$$

then $\mathbf{k}$ cannot be a maximal vector.

Lemma 4. *If, for a particular pair of quantities m and n ($m|n$) the set K^* contains the only element $\mathbf{k}^*$, then (i) (1.6) and (1.9) hold, (ii) in (1.3) and (1.8) equalities stand only if $f(x)$ is a multiple of (1.7) or (1.10), respectively.*

Lemma 5. If $m|n$, $\mathbf{k} \in K^*$ then $\min_{r=1, 2, \dots, m} k_r \equiv 0 \pmod{m}$.

Lemma 6. If for a particular $m > 2$ the quantity

$$(2.8) \quad H(m) = \max_{l=1, 2, \dots, m-1} \left\{ -m + l + 4 \sum_{r=1}^{\left[\frac{m-1}{2} \right]} \max \left\{ E_l \left(\frac{2\pi}{m} r \right), 0 \right\} \right\}$$

satisfies the inequality

$$(2.9) \quad H(m) < \frac{11}{3}$$

and n is any multiple of m , then the set K^* consists of the only vector $\mathbf{k}^*$.

Lemma 7. If $H(n) < 0$ for a particular n , then for this n the solution of the Problem of the Introduction is that given in § 1.

§ 3. Proof of Lemmas 1–5 and of the Theorems

If, by virtue of the supposition of Lemma 1 one has for some m and n ($m|n$) and for any $f \in \pi'_n$

$$\frac{1}{m} \left| \sum_{r=1}^m \sum_{v=1}^{k_r} \left(a_v \cos v \frac{2\pi}{m} r + b_v \sin v \frac{2\pi}{m} r \right) \right| \leq \sqrt{\frac{n}{m}} \left\{ \sum_{v=1}^n (|a_v|^2 + |b_v|^2) \right\}^{\frac{1}{2}}$$

with equality only if $f = c(\cos mx + \cos 2mx + \dots + \cos nx)$ and $k_1 = k_2 = \dots = k_m = n$, then for any a_0 we have

$$\begin{aligned} & \frac{1}{m} \left| \sum_{r=1}^m \left\{ \frac{a_0}{2} + \sum_{v=1}^{k_r} \left(a_v \cos v \frac{2\pi}{m} r + b_v \sin v \frac{2\pi}{m} r \right) \right\} \right| \leq \\ & \leq \left| \frac{a_0}{2} \right| + \frac{1}{m} \left| \sum_{r=1}^m \sum_{v=1}^{k_r} \left(a_v \cos v \frac{2\pi}{m} r + b_v \sin v \frac{2\pi}{m} r \right) \right| \leq \\ & \leq \frac{1}{\sqrt{2}} \left| \frac{a_0}{\sqrt{2}} \right| + \sqrt{\frac{n}{m}} \left\{ \sum_{v=1}^n (|a_v|^2 + |b_v|^2) \right\}^{\frac{1}{2}} \leq \sqrt{\frac{1}{2} + \frac{n}{m}} \left\{ \frac{|a_0|^2}{2} + \sum_{v=1}^n (|a_v|^2 + |b_v|^2) \right\}^{\frac{1}{2}} \end{aligned}$$

by Cauchy's inequality, with equality everywhere only if $k_1 = k_2 = \dots = k_m = n$ and $f = c(\frac{1}{2} + \cos mx + \cos 2mx + \dots + \cos nx)$. This proves Lemma 1.

Lemma 2 can be proved with the method of Kolmogoroff and Seliverstoff ([3]) just as it was done in a similar case in § 4 of Part I of this paper; we do not repeat it here.

To prove Lemma 3 we note that one has in this particular case with the notation of (2.3)

$$(3.1) \quad e_{pp} = k_p, e_{pq} = \sum_{z=1}^{a_{pq}m} \cos z \frac{2\pi}{m} (p-q) = 0 \quad (p \neq q)$$

where the integer a_{pq} is defined by $\min(k_p, k_q) = a_{pq}m$ and so

$$(3.2) \quad E(\mathbf{k}) = \sum_{p=1}^m k_p \equiv \sum_{p=1}^m n = E(\mathbf{k}^*)$$

with equality only if $\mathbf{k} = \mathbf{k}^*$.

In view of the foregoing and of formula (1.5) the first half of the statement of Lemma 4 is evident. As to its second half we treat the case $f \in \pi_n$ only. Then we have

$$\frac{1}{m} \left| \sum_{r=1}^m s_{k_r} \left(\frac{2\pi}{m} r \right) \right| < \sqrt{\frac{1}{2} + \frac{n}{m}} \|f\|$$

if $\mathbf{k}$ is an admissible vector different from $\mathbf{k}^*$ and $f \in \pi_n$.

If, however $\mathbf{k} = \mathbf{k}^*$ one has by Cauchy's inequality

$$\begin{aligned} \frac{1}{m} \left| \sum_{r=1}^m s_{k_r} \left(\frac{2\pi}{m} r \right) \right| &= \frac{1}{m} \left| \sum_{r=1}^m f \left(\frac{2\pi}{m} r \right) \right| = \left| \frac{a_0}{2} + a_m + a_{2m} + \dots + a_n \right| \equiv \\ &\equiv \sqrt{\frac{1}{2} + \frac{n}{m}} \left\{ \frac{|a_0|^2}{2} + |a_m|^2 + |a_{2m}|^2 + \dots + |a_n|^2 \right\}^{\frac{1}{2}} \equiv \sqrt{\frac{1}{2} + \frac{n}{m}} \|f\| \end{aligned}$$

and equality stands on both places only if $a_0 = a_m = a_{2m} = \dots = a_n$, $a_v = 0$ for $v \not\equiv 0 \pmod{m}$, $b_v = 0$ for each v .

The case $f \in \pi'_n$ can be dealt with similarly.

Lemma 5 is an obvious version of Theorem 5 of Part I of this paper. Its proof is incorporated alongside with similar statements in § 6. (Formula (6.3)).

Theorem 1 may be deduced from Lemmas 2 and 5. Lemma 5 states in the particular case $m = n$ that if $\mathbf{k} \in K^*$ and the vector $\mathbf{k}^*$ defined by (2.7) is not contained in K^* then at least one element of $\mathbf{k}$ vanishes. We shall show that the assumption $\mathbf{k}^* \notin K^*$ is in contradiction with the supposition of Theorem 1, hence $\mathbf{k}^* \in K^*$ and $\lambda'_n(1, 2, \dots, n) = [E(\mathbf{k}^*)]^{\frac{1}{2}} = n$.

Indeed if $\mathbf{k} \notin K^*$ and $\mathbf{k}^* \in K^*$ then we may suppose without loss of generality that the last element of the vector $\mathbf{k}^* = \{k_1^*, \dots, k_{n-1}^*, k_n^*\}$ vanishes: $k_n^* = 0$. Then by Lemma 2 and by the definition (1.11) of $\lambda'_n(1, 2, \dots, \mu)$

$$\begin{aligned} [\lambda'_n(1, 2, \dots, n)]^2 &= \sum_{p=1}^n \sum_{q=1}^n E_{\min(k_p^*, k_q^*)} \left(\frac{2\pi}{n} (p-q) \right) = \\ &= \sum_{p=1}^{n-1} \sum_{q=1}^{n-1} E_{\min(k_p^*, k_q^*)} \left(\frac{2\pi}{n} (p-q) \right) \equiv [\lambda'_n(1, 2, \dots, n-1)]^2 \end{aligned}$$

which contradicts the supposition of Theorem 1.

Finally turning to the proof of Theorem 2 let $f_\mu(x)$ be an extremal function of (1.11) in the case $r_\mu = \mu$ ($\mu = 1, 2, \dots, \mu$).

(a) If any partial sum (up to the n 'th) of $f_{n-1}(x)$ is positive on the place $x=2\pi$, then $\lambda'_n(1, 2, \dots, n)$ is by definition greater than $\lambda'_n(1, 2, \dots, n-1)$:

$$\begin{aligned}\lambda'_n(1, 2, \dots, n-1) &= \sum_{\kappa=1}^{n-1} \max_k s_k \left(\frac{2\pi}{n} \kappa; f_{n-1} \right) < \\ &< \sum_{\kappa=1}^n \max_k s_k \left(\frac{2\pi}{n} \kappa; f_{n-1} \right) \equiv \lambda'_n(1, 2, \dots, n).\end{aligned}$$

If however each of these partial sums is non-positive, then $\max_k s_k(2\pi, f_{n-1}) = s_0(2\pi, f_{n-1}) = 0$ and by supposition (a) of Theorem 2 $f_n(x) \neq f_{n-1}(x)$, hence

$$\lambda'_n(1, 2, \dots, n-1) = \sum_{\kappa=1}^n \max_k s_k \left(\frac{2\pi}{n} \kappa; f_{n-1} \right) < \lambda'_n(1, 2, \dots, n).$$

In both cases we have $\lambda'_n(1, 2, \dots, n-1) < \lambda'_n(1, 2, \dots, n)$ and Theorem 1 can be applied.

Now from the definition of $\lambda'_n(1, 2, \dots, n)$ it is clear that if

$$(3.3) \quad f_n(x) = \sum_{v=1}^n (a_v^* \cos vx + b_v^* \sin vx)$$

is an extremal function, then

$$(3.4) \quad \begin{aligned}f_n \left(x - \frac{2\pi}{n} \right) &= \\ &= \sum_{v=1}^n \left\{ \left(a_v^* \cos \frac{2\pi}{n} v - b_v^* \sin \frac{2\pi}{n} v \right) \cos vx + \left(b_v^* \cos \frac{2\pi}{n} v + a_v^* \sin \frac{2\pi}{n} v \right) \sin vx \right\}\end{aligned}$$

must also be an extremal.

(b) If, by supposition $f_n(x)$ and $f_n \left(x - \frac{2\pi}{n} \right)$ are both even functions, we have by (3.3) and (3.4)

$$b_v^* = 0, \quad a_v^* \sin \frac{2\pi}{n} v = 0 \quad (v = 1, 2, \dots, n)$$

hence for odd n 's $a_v^* = 0$ for $v = 1, 2, \dots, n-1$. Finally from $\|f_n(x)\| = 1$ we have $f_n(x) = \pm \cos nx$. Since only the positive sign yields a maximum, we can apply the first part of Theorem 2.

If, however, n is even we are led to the conclusion that any extremal must be of the form $f_n(x) = a_n^* \cos nx + a_{n/2}^* \cos nx/2$. Now we have to distinguish several cases.

Case 1: $a_n^* \leq 0$. Then

$$\begin{aligned}&\sum_{\kappa=1}^n \max_{k=0, \dots, n} s_k \left(\frac{2\pi}{n} \kappa; a_{n/2}^* \cos \frac{n}{2} x + a_n^* \cos nx \right) = \\ &= \sum_{\kappa=1}^n \max_{k=0, \dots, n/2} s_k \left(\frac{2\pi}{n} \kappa; a_{n/2}^* \cos \frac{n}{2} x \right) = \frac{n}{2} |a_{n/2}^*| < n\end{aligned}$$

since $|a_{n/2}^*| \leq 1$.

Case 2. 1: $a_n^* > |a_{n/2}^*| \geq 0$. Then

$$\sum_{\kappa=1}^n \max_{k=0, \dots, n} s_k \left(\frac{2\pi}{n} \kappa; a_{n/2}^* \cos \frac{n}{2} x + a_n^* \cos nx \right) = \sum_{\kappa=1}^n [a_n^* + (-1)^\kappa a_{n/2}^*] = na_n^* \leq n$$

with equality only if $a_n^* = 1, a_{n/2}^* = 0$ since $a_n^{*2} + a_{n/2}^{*2} = 1$.

Case 2. 2: $|a_{n/2}^*| \geq a_n^* > 0$. Now

$$\begin{aligned} \sum_{\kappa=1}^n \max_{k=0, \dots, n} s_k \left(\frac{2\pi}{n} \kappa; a_{n/2}^* \cos \frac{n}{2} x + a_n^* \cos nx \right) &= \sum_{f_n \left(\frac{2\pi}{n} j \right) \geq 0} f_n \left(\frac{2\pi}{n} j \right) = \\ &= \frac{n}{2} (a_n^* + |a_{n/2}^*|) \leq n \sqrt{\frac{a_n^{*2} + a_{n/2}^{*2}}{2}} < n \quad (j = 1, 2, \dots, n). \end{aligned}$$

Summing up the different cases the maximum is attained only for $f_n(x) = \cos nx$ and we are led again to the first part of Theorem 2.

(c) Turning to the last part of this theorem if $f_n(x)$ is unique for some n , then

$f_n(x) = f_n \left(x - \frac{2\pi}{n} \right)$ and from (3.3) and (3.4) one has

$$a_v^* \cos \frac{2\pi}{n} v - b_v^* \sin \frac{2\pi}{n} v = a_v^*$$

$$a_v^* \sin \frac{2\pi}{n} v + b_v^* \cos \frac{2\pi}{n} v = b_v^*.$$

This system has only the trivial solution, save if $v = n$, i. e. the extremal function is necessarily of the form $f_n(x) = a_n^* \cos nx + b_n^* \sin nx$ ($a_n^{*2} + b_n^{*2} = 1$). Now

$$\sum_{\kappa=1}^n \max_{k=0, 1, \dots, n} s_k \left(\frac{2\pi}{n} \kappa; a_n^* \cos nx + b_n^* \sin nx \right) = \begin{cases} 0, & \text{if } a_n^* \leq 0 \\ na_n^*, & \text{if } a_n^* \geq 0 \end{cases}$$

and the left hand side is equal to n only if $a_n^* = 1, b_n^* = 0$, in all other cases it is less than n , hence $f_n(x) = \cos nx$, and $\lambda_n'(1, 2, \dots, n) = n$.

§ 4. A generalization of the problem of the introduction

Our next aim is to deal with the case $m \leq 38$ in Statement (C') of the Introduction and to solve simultaneously the problem given in § 1.

The solution of both problems is contained in the following

Lemma 8. If $m|n, k_r = 0, 1, \dots, n$ ($r = 1, 2, \dots, m$), $m \leq 38, 1 \leq r_1 < r_2 < \dots < r_\mu \leq m$ (r_κ integer for $\kappa = 1, 2, \dots, \mu$) and $\lambda_n^{(m)}(r_1, r_2, \dots, r_\mu)$ is the least positive quantity for which

$$(4.1) \quad \left| \sum_{\kappa=1}^{\mu} s_{k_{r_\kappa}} \left(\frac{2\pi}{m} r_\kappa; f \right) \right| \leq \lambda_n^{(m)}(r_1, r_2, \dots, r_\mu) \quad (f \in \pi_n', \|f\| = 1)$$

for any set $k_{r_1}, k_{r_2}, \dots, k_{r_\mu}$ then

$$(4.2) \quad \lambda_n^{(m)}(r_1, r_2, \dots, r_\mu) = \sqrt{n\mu}.$$

Equality occurs only if $k_{r_1} = k_{r_2} = \dots = k_{r_\mu} = n$ and $f(x)$ is equal to

$$(4.3) \quad \frac{1}{\sqrt{n\mu}} \sum_{\kappa=1}^{\mu} E_n \left(x - \frac{2\pi}{m} r_{\kappa} \right)$$

multiplied by a constant of modulus 1.

Hence statement (C') follows for $m \leq 38$ by taking $\mu = m$, $r_{\kappa} = \kappa$ ($\kappa = 1, 2, \dots, m$) since

$$\frac{1}{m} \sum_{r=1}^m E_n \left(x - \frac{2\pi}{m} r \right) = \frac{1}{m} \sum_{v=1}^n \sum_{r=1}^m \cos v \left(x - \frac{2\pi}{m} r \right) = \cos mx + \cos 2mx + \dots + \cos nx.$$

On the other hand, one has by Lemma 2, that if Lemma 8 holds, then with the notations employed there,

$$(4.4) \quad \sum_{\kappa=1}^{\mu} s_{k_{r_{\kappa}}} \left(\frac{2\pi}{m} r_{\kappa}; f \right) \leq \sqrt{n\mu} \quad (f \in \pi_n'')$$

and equality occurs only if $k_{r_1} = k_{r_2} = \dots = k_{r_\mu} = n$ and $f(x)$ is equal to the function (4.3) or with the notations of Lemma 2

$$\lambda''(x, \mathbf{k}) = \sqrt{n\mu} \quad \text{if} \quad x = \left\{ \frac{2\pi}{m} r_1, \dots, \frac{2\pi}{m} r_{\mu} \right\}, \quad \mathbf{k} = \left\{ \frac{1}{n}, \frac{2}{n}, \dots, \frac{\mu}{n} \right\}.$$

In the special case $n = m$ the inequality (4.4) is equivalent to (1.11)–(1.12). Since for $x \not\equiv 0 \pmod{2\pi}$

$$E_n(x) = \frac{\sin \left(n + \frac{1}{2} \right) x}{2 \sin \frac{x}{2}} - \frac{1}{2},$$

one has

$$E_n(x) = \begin{cases} n & \text{if } x = 0 \\ 0 & \text{if } x = \frac{2\pi}{n} r \quad (r = 1, 2, \dots, n-1) \\ -1 & \text{if } x = \frac{\pi}{n} (2r+1) \quad (r = 0, 1, \dots, n-1) \end{cases}$$

further

$$E_n \left(x - \frac{2\pi}{n} r \right) = \sum_{v=1}^{n-1} \cos v \left(x - \frac{2\pi}{n} r \right) + \cos nx,$$

therefore in our special case the function (4.3), the only function in π_n'' for which (4.4) is satisfied with the sign of equality, is by virtue of (2.6) the one characterized by conditions (i)–(iv) in § 1.

§ 5. Proof of Lemma 6 in the case $H(m) < 0$ and of Lemmas 7 and 8

In order to prove Lemma 6 we shall give an upper estimation for the quantities $E(\mathbf{k}) = \sum e_{pq}$ (see 2. 3). Suppose that γ_l of the elements of the vector $\mathbf{k}$ satisfy the congruences $k_p \equiv l \pmod{m}$ ($l=0, 1, \dots, m-1$) so that with $\gamma = \gamma_0$

$$(5.1) \quad \gamma + \gamma_1 + \gamma_2 + \dots + \gamma_{m-1} = m$$

and the elements $k_{p_1}, k_{p_2}, \dots, k_{p_\gamma}$ should be divisible by m .

Putting $h = n/m$ we have

$$\sum_{p=1}^m e_{pp} = \sum_{p=1}^m k_p \equiv \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l \{(h-1)m + l\}.$$

On the other hand if $p \neq q$ and $x_r = 2\pi r/m$, $e_{pq} = E_{\min(k_p, k_q)}(x_p - x_q) \equiv \max\{E_{k_p}(x_p - x_q), 0\} + \max\{E_{k_q}(x_p - x_q), 0\}$ and we can estimate the sum of the off-diagonal elements of $\mathbf{E}(\mathbf{k})$:

$$\begin{aligned} \sum_{p=1}^m \sum_{q \neq p} e_{pq} &\equiv \sum_{p=1}^m \sum_{q \neq p} \max\{E_{k_p}(x_p - x_q), 0\} + \sum_{q=1}^m \sum_{p \neq q} \max\{E_{k_q}(x_q - x_p), 0\} = \\ &= 2 \sum_{p=1}^m \sum_{r=1}^{m-1} \max\{E_{k_p}(x_r), 0\} = 4 \sum_{p=1}^m F(k_p, m) \end{aligned}$$

where for $m > 2$

$$F(k_p, m) = \frac{1}{2} \sum_{r=1}^{m-1} \max\{E_{k_p}(x_r), 0\} = \sum_{r=1}^{\left[\frac{m-1}{2}\right]} \max\{E_{k_p}(x_r), 0\}$$

since

$$E_{k_p}(x_r) = E_{k_p}(x_{m-r})$$

and in the case of an even m

$$E_{k_p}(x_{m/2}) = \cos \pi + \cos 2\pi + \dots + \cos k_p \pi \equiv 0.$$

We remark that

$$(5.2) \quad F(k_p, m) = F(k_p + m, m) \text{ and } F(0, m) = 0.$$

Summing up we have

$$\begin{aligned} (5.3) \quad E(\mathbf{k}) &\equiv \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l \{(h-1)m + l\} + 4 \sum_{l=1}^{m-1} \gamma_l F(l, m) \equiv \\ &\equiv \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l n + \sum_{l=1}^{m-1} \gamma_l H(m) = M(\mathbf{k}) \end{aligned}$$

by the definition (2. 8) of $H(m)$.

Table of the values of the function $H(m)$ for $33 \leq m \leq 60$
(If $3m \leq 32$, then $H(m) = -1$.) [2]

m	$H(m)$	m	$H(m)$	m	$H(m)$	m	$H(m)$
33	-0,8370	40	0,3252	47	1,3644	54	4,1527
34	-0,8346	41	0,4410	48	2,5628	55	3,9579
35	-0,5882	42	1,0522	49	2,2586	56	3,9382
36	-0,1857	43	0,9025	50	2,5996	57	4,9962
37	-0,2992	44	0,8997	51	3,2941	58	4,5419
38	-0,4837	45	1,8167	52	2,9881	59	4,5401
39	0,4421	46	1,5694	53	2,8764	60	5,8271

The accompanying table of the values of $H(m)$ [2] shows that $H(m) < 0$, if $2 < m \leq 38$ and so it follows immediately that

$$(5.4) \quad E(\mathbf{k}) \leq M(\mathbf{k}) < \sum_{i=1}^{\gamma} k_{p_i} + \sum_{l=1}^{m-1} \gamma_l n \leq \gamma n + \sum_{l=1}^{m-1} \gamma_l n = mn$$

if $\gamma_1 + \gamma_2 + \dots + \gamma_{m-1} > 0$ and $2 < m \leq 38$.

On the other hand if $m \leq 38$ and $\gamma_1 + \gamma_2 + \dots + \gamma_{m-1} = 0$, then $\gamma = m$ and Lemma 3 can be applied:

$$(5.5) \quad E(\mathbf{k}) \leq E(\mathbf{k}^*) = M(\mathbf{k}^*) = mn$$

with equality only if $\mathbf{k} = \mathbf{k}^*$.

So we found that if $2 < m \leq 38$ the set K^* contains the only element $\mathbf{k}^*$ and Lemma 6 is proved for these m 's. Using Lemmas 4 and 1 we see that Conjecture (C) and Statement (C') are also proved for these m 's.

Turning to Lemma 7 we regard it as a consequence of Lemma 8 in the particular case $m = n \leq 38$ (cfr. the end of § 4) since we know that $H(n) < 0$ for $n \leq 38$ and shall prove the more general Lemma 8.

Lemma 8 was already proved in the particular case $\mu = m$ ($r_1 = 1, r_2 = 2, \dots, r_m = m$) for we have shown that if $m \leq 38, m|n$, then

$$E(\mathbf{k}) \leq M(\mathbf{k}), M(\mathbf{k}) < mn \quad \text{if } \mathbf{k} \in K \quad \text{and} \quad \mathbf{k} \neq \mathbf{k}^*$$

and

$$[\lambda_n^{(m)}(1, 2, \dots, m)]^2 = \max_{\mathbf{k} \in K} E(\mathbf{k}) = E(\mathbf{k}^*) = mn.$$

Since K^* contains the only element $\mathbf{k}^*$ there is a unique extremal function which can be found by applying Lemma 2.

Now we turn to the case $\mu < m$. To any sequence of μ numbers $k_{r_1}, k_{r_2}, \dots, k_{r_\mu}$ we adjoin two m -dimensional vectors $\mathbf{k}_r$ and $\mathbf{k}^r$ of the set K . The r_α 'th components of both of these vectors will be equal to k_{r_α} ($\alpha = 1, 2, \dots, \mu$) whereas if q is not equal to any of the numbers $r_1, r_2, \dots, r_\mu$ then the q 'th component of $\mathbf{k}_r$ will be 0 and the same component of $\mathbf{k}^r$ will be n . The set of all vectors $\mathbf{k}_r \in K$ will be denoted by K_r , i. e. K_r is the set of all admissible vectors whose q 'th components are 0 if q is not equal to any of the numbers $r_1, r_2, \dots, r_\mu$.

We have by Lemma 2

$$(5.6) \quad [\lambda_n^{(m)}(r_1, \dots, r_\mu)]^2 = \max_{\substack{k_{r_\sigma} = 0, 1, \dots, n \\ \sigma = 1, 2, \dots, \mu}} \sum_{\sigma=1}^{\mu} \sum_{\tau=1}^{\mu} E_{\min(k_{r_\sigma}, k_{r_\tau})} \left(\frac{2\pi}{m} (r_\sigma - r_\tau) \right) =$$

$$= \max_{\mathbf{k} \in K_r} \sum_{p=1}^m \sum_{q=1}^m E_{\min(k_p, k_q)} \left(\frac{2\pi}{m} (p - q) \right) = \max_{\mathbf{k} \in K_r} E(\mathbf{k})$$

since in the case $\mathbf{k} \in K_r$

$$E_{\min(k_p, k_q)} \left(\frac{2\pi}{m} (p - q) \right) = E_0 \left(\frac{2\pi}{m} (p - q) \right) = 0$$

if at least one of the quantities p and q is not contained in the set $r_1, r_2, \dots, r_\mu$.

Further, it follows from (5.5) and (5.3) that for $m \leq 38, m|n$

$$(5.7) \quad mn \equiv M(\mathbf{k}^r) = M(\mathbf{k}_r) + (m - \mu)n$$

with equality only if $\mathbf{k}^r = \mathbf{k}^*$ i. e. the components of $\mathbf{k}^r$ are either n 's or 0 's. This last vector will be denoted by $\mathbf{k}_r^*$ and one has by (3.2)

$$(5.8) \quad E(\mathbf{k}_r^*) = \mu n.$$

On the other hand from (5.7) and (5.3) we have that

$$\mu n > M(\mathbf{k}_r) \geq E(\mathbf{k}_r),$$

if $\mathbf{k}_r \in K_r, \mathbf{k}_r \neq \mathbf{k}_r^*$. Hence by (5.6)

$$(5.9) \quad [\lambda_n^{(m)}(r_1, r_2, \dots, r_\mu)]^2 \geq E(\mathbf{k}) \quad (\mathbf{k} \in K_r)$$

and equality stands only if $\mathbf{k} = \mathbf{k}_r^*$. In view of (5.8) this is the first part of the statement of Lemma 8. Its second part, formula (4.3), is derived from the unicity property of the vector $\mathbf{k}_r^*$ in connection with inequality (5.9), and from the statement of Lemma 2 regarding the extremal function. With this, we have got the solution of the Problem of § 1, too.

§ 6. Proof of Lemma 6 in the case $0 \leq H(m) \leq 11/3$

In dealing with the case $0 \leq H(m) \leq 11/3$ (this contains by virtue of the Table the cases $39 \leq m \leq 53$), we shall need the estimation

$$(6.1) \quad E_l(x) \geq -C(l-1) - 1 \quad \text{with} \quad C = \frac{1 - (3 - 2\sqrt{2})^2}{\sqrt{2}\pi} = 0.21845.$$

This estimation is obviously valid if $l=1$. Hence we may restrict ourselves to the cases $l>1$. Obviously it is sufficient to regard the interval $(0, \pi)$ and in this

only the subintervals

$$\left(\frac{\pi}{l'}, \frac{2\pi}{l'}\right), \left(\frac{3\pi}{l'}, \frac{4\pi}{l'}\right), \left(\frac{5\pi}{l'}, \frac{6\pi}{l'}\right), \dots$$

where $l' = l + \frac{1}{2}$. Now if x lies in one of these subintervals but not in the first one, then

$$E_l(x) = \frac{\sin l' x}{2 \sin \frac{1}{2} x} - \frac{1}{2} > \frac{\sin l' \left(x - \frac{2\pi}{l'}\right)}{2 \sin \frac{1}{2} \left(x - \frac{2\pi}{l'}\right)} - \frac{1}{2} = E_l \left(x - \frac{2\pi}{l'}\right).$$

Hence the place x_0 of the absolute minimum of $E_l(x)$ in $(0, \pi)$ lies in $(\pi/l', 2\pi/l')$. Moreover it is easily seen that

$$\frac{d}{dx} E_l(x) > 0 \quad \text{if} \quad \frac{3\pi}{2l'} \leq x \leq \frac{2\pi}{l'}$$

and so $\pi/l' < x_0 < 3\pi/(2l')$.

We estimate $E_l(x)$ in this last interval as follows:

$$\begin{aligned} E_l(x) &= -\left\{ \cos \left(l' x - \frac{3}{2} \pi \right) \right\} \left\{ 2 \sin \frac{x}{2} \right\}^{-1} - \frac{1}{2} \equiv \\ &\equiv -\left\{ 1 - \left(\frac{2}{\pi} \right)^2 \left(l' x - \frac{3}{2} \pi \right)^2 \right\} \left\{ x \frac{\sin 3\pi/(4l')}{3\pi/(4l')} \right\}^{-1} - \frac{1}{2} = E_l^*(x). \end{aligned}$$

Here we used the inequalities

$$\cos \alpha < 1 - \left(\frac{2}{\pi} \right)^2 \alpha^2 \quad \text{if} \quad 0 < \alpha < \frac{\pi}{2} \quad \text{and} \quad \frac{\sin \beta}{\beta} > \frac{\sin \gamma}{\gamma} \quad \text{if} \quad 0 < \beta < \gamma < \frac{\pi}{2}. *$$

The place and value of the minimum of the function $E_l^*(x)$ can be calculated directly and so we have

$$E_l(x) \geq \min E_l^*(x) = -C \frac{3\pi/4}{\sin 3\pi/(4l')} - \frac{1}{2} > -C(l+1) - \frac{1}{2} > -C(l-1) - 1$$

since for $l \geq 2$

$$\sin \frac{3\pi}{4l'} > \frac{3\pi}{4l'} \left[1 - \frac{1}{6} \left(\frac{3\pi}{4l'} \right)^2 \right]^2 > \frac{3\pi}{4(l+1)}.$$

*) The second of these inequalities is obvious. If one puts in it $\beta = \alpha/2$, $\gamma = \pi/4$ one has

$$\frac{\sin \alpha/2}{\alpha/2} > \frac{\sin \pi/4}{\pi/4} \quad \text{if} \quad 0 < \alpha < \frac{\pi}{2}.$$

This is equivalent to the first inequality by virtue of the relation $1 - \cos \alpha = 2 \sin^2 \alpha/2$.

Let now $\alpha_1, \alpha_2, \dots, \alpha_m$ be such a permutation of the numbers $1, 2, \dots, m$ that the coordinates of the m dimensional vector $\mathbf{k}$ satisfy the inequalities

$$k_{\alpha_1} \leq k_{\alpha_2} \leq \dots \leq k_{\alpha_m}.$$

Let further be

$$k_{\alpha_i} = k'_{\alpha_i} + k''_{\alpha_i} \quad (i=1, 2, \dots, m)$$

where k'_{α_i} is the largest multiple of m not exceeding k_{α_i} .

We define the vector $\mathbf{k}^{(j)}$ as follows. All its coordinates are equal to the corresponding coordinates of $\mathbf{k}$, save the α_1 'th, α_2 'th, $\dots$, α_j 'th: these are $k'_{\alpha_1}, k'_{\alpha_2}, \dots, k'_{\alpha_j}$, respectively.

Then by (2.1)

$$(6.2) \quad E(\mathbf{k}) - E(\mathbf{k}^{(j)}) = -k''_{\alpha_1} + 2 \sum_{r=1}^m \left\{ E_{k_{\alpha_1}} \left(\frac{2\pi}{m} r \right) - E_{k'_{\alpha_1}} \left(\frac{2\pi}{m} r \right) \right\} - \\ - k''_{\alpha_2} + 2 \sum_{\substack{r \neq \alpha_2 - \alpha_1 \\ r \neq \alpha_2 - \alpha_2}} \left\{ E_{k_{\alpha_2}} \left(\frac{2\pi}{m} r \right) - E_{k'_{\alpha_2}} \left(\frac{2\pi}{m} r \right) \right\} - \\ \dots \dots \dots - k''_{\alpha_j} + 2 \sum_{\substack{r \neq \alpha_j - \alpha_1 \\ r \neq \alpha_j - \alpha_2 \\ \dots \\ r \neq \alpha_j - \alpha_{j-1}}} \left\{ E_{k_{\alpha_j}} \left(\frac{2\pi}{m} r \right) - E_{k'_{\alpha_j}} \left(\frac{2\pi}{m} r \right) \right\}.$$

Let us denote by S_i the i 'th row of the right hand side of the last equality.

Then

$$S_i = -k''_{\alpha_i} + 2 \sum_{r=1}^m \left\{ \cos \frac{2\pi}{m} r + \cos 2 \cdot \frac{2\pi}{m} r + \dots + \cos k'_{\alpha_i} \frac{2\pi}{m} r \right\} - \\ - 2 \sum_{\mu=1}^{i-1} \left\{ \cos \frac{2\pi}{m} (\alpha_i - \alpha_\mu) + \cos 2 \cdot \frac{2\pi}{m} (\alpha_i - \alpha_\mu) + \dots + \cos k'_{\alpha_i} \frac{2\pi}{m} (\alpha_i - \alpha_\mu) \right\} = \\ = -k''_{\alpha_i} - 2 \sum_{\mu=1}^{i-1} E_{k'_{\alpha_i}} (\alpha_i - \alpha_\mu) < -k''_{\alpha_i} + 2 \sum_{\mu=1}^{i-1} \{ C(k'_{\alpha_i} - 1) + 1 \}$$

by (6.1). From this follow

$$S_1 = -k''_{\alpha_1} \leq 0 \\ S_2 = (2C - 1)k''_{\alpha_2} + 2(1 - C) \leq 2(1 - C) \\ S_3 \leq (4C - 1)k''_{\alpha_3} + 4(1 - C) \leq 4(1 - C) \\ S_4 \leq (6C - 1)(m - 2) + 5.$$

Hence we have

$$(6.3) \quad E(\mathbf{k}) \leq E(\mathbf{k}^{(1)})$$

$$(6.4) \quad E(\mathbf{k}) \leq E(\mathbf{k}^{(2)}) + 2(1 - C) < E(\mathbf{k}^{(2)}) + 2$$

$$(6.5) \quad E(\mathbf{k}) \leq E(\mathbf{k}^{(3)}) + 6(1 - C) < E(\mathbf{k}^{(3)}) + 5$$

$$(6.6) \quad E(\mathbf{k}) \leq E(\mathbf{k}^{(4)}) + (6C - 1)(m - 2) + 5 + 6(1 - C) < E(\mathbf{k}^{(4)}) + \frac{m}{3} + 10.$$

We divide now the set K into the following five subsets.

- (i) $K^{(1)}$: it contains the only element $\mathbf{k}^*$ defined by (2. 7).
- (ii) $K^{(2)}$: its vectors are characterized by $k_{x_1} < n, k_{x_2} = \dots = k_{x_m} = n$.
- (iii) $K^{(3)}$: its vectors are characterized by $k_{x_1} \leq k_{x_2} < n, k_{x_3} = \dots = k_{x_m} = n$.
- (iv) $K^{(4)}$: its vectors are characterized by $k_{x_1} \leq k_{x_2} \leq k_{x_3} < n, k_{x_4} = \dots = k_{x_m} = n$.
- (v) $K^{(5)}$: its vectors are characterized by $k_{x_1} \leq k_{x_2} \leq k_{x_3} \leq k_{x_4} < n$.

Clearly $K = K^{(1)} \dot{+} K^{(2)} \dot{+} \dots \dot{+} K^{(5)}$.

Our purpose is now to show that if $m|n$ and $m \geq 5$, then the sets $K^{(2)}, K^{(3)}$ and $K^{(4)}$ contain no maximal vectors and if $m|n, H(m) < 11/3$ the subset $K^{(5)}$, too, has no maximal element.

Suppose $\mathbf{k} \in K^{(2)}$. Then by (6. 3) and Lemma 3 $E(\mathbf{k}) < E(\mathbf{k}^*)$.

Again, if $\mathbf{k} \in K^{(3)}$, then

$$\begin{aligned} E(\mathbf{k}) < E(\mathbf{k}^{(2)}) + 2 &= k'_1 + k'_2 + (m-2)n + 2 \leq \\ &\leq 2(n-m) + (m-2)n + 2 = (n-2)m + 2 < E(\mathbf{k}^*) \end{aligned}$$

and similarly if $\mathbf{k} \in K^{(4)}$, then

$$E(\mathbf{k}) \leq E(\mathbf{k}^{(3)}) + 5 < (n-3)m + 5 < E(\mathbf{k}^*).$$

Finally if $\mathbf{k} \in K^{(5)}, m \geq 5$, then by (6. 6) and (5. 3)

$$\begin{aligned} E(\mathbf{k}) &< E(\mathbf{k}^{(4)}) + \frac{m}{3} + 10 \\ &\leq \gamma n - 4m + (m-\gamma)n + \sum_{l=1}^{m-1} \gamma_l H(m) + \frac{m}{3} + 10 \\ &\leq mn - 4m + (m-4)H(m) + \frac{m}{3} + 10 \\ &= mn + \left\{ H(m) - \frac{11}{3} \right\} (m-3) - H(m) - 1 \end{aligned}$$

where $\gamma, \gamma_1, \dots, \gamma_{m-1}$ are the numbers defined in Section 5, referred to the vector $\mathbf{k}^{(4)}$.

The last row is less than $E(\mathbf{k}^*) = mn$, if $-1 \leq H(m) \leq 11/3$ i. e. by the Table if $2 < m \leq 53$, and so via Lemma 4 Statement (C') of § 1 is verified, too, for these m 's.

§ 7. Proof of Statement (C') in the cases $m=1$ and $m=2$

The remaining cases $m=1$ and $m=2$ can be treated quite simply. In the case $m=1, k_1 \leq n, f \in \pi'_n$

$$\begin{aligned} |s_{k_1}(2\pi)| &= |a_1 + a_2 + \dots + a_{k_1}| \leq \sqrt{k_1} \{ |a_1|^2 + |a_2|^2 + \dots + |a_{k_1}|^2 \}^{\frac{1}{2}} \leq \\ &\leq \sqrt{n} \left\{ \sum_{v=1}^n (|a_v|^2 + |b_v|^2) \right\}^{\frac{1}{2}} \end{aligned}$$

and equality stands on both places only if $k_1 = n, a_1 = a_2 = \dots = a_n, b_1 = b_2 = \dots = b_n = 0$.

In the case $m=2$ and $k_1 \leq k_2 \leq n$, say,

$$\begin{aligned} & \frac{1}{2} |s_{k_1}(\pi) + s_{k_2}(2\pi)| = \\ & = \left| a_2 + a_4 + \dots + \frac{1+(-1)^{k_1}}{2} a_{k_1} + \frac{1}{2} a_{k_1+1} + \frac{1}{2} a_{k_1+2} + \dots + \frac{1}{2} a_{k_2} \right| \leq \\ & \leq \left\{ \left\lfloor \frac{k_1}{2} \right\rfloor + \frac{1}{2^2} (k_2 - k_1) \right\}^{\frac{1}{2}} \{ |a_2|^2 + |a_4|^2 + \dots + |a_{2\lfloor k_1/2 \rfloor}|^2 + \\ & + |a_{k_1+1}|^2 + |a_{k_1+2}|^2 + \dots + |a_{k_2}|^2 \}^{\frac{1}{2}} \leq \left(\frac{k_1 + k_2}{4} \right)^{\frac{1}{2}} \|f\| \leq \sqrt{\frac{n}{2}} \|f\| \end{aligned}$$

with equality everywhere only if $k_1 = k_2 = n$, n even, and $a_v = a_2$ if v is even, $a_v = 0$ if v is odd and $b_v = 0$ for each v . *)

Acknowledgement

The table of the values of $H(m)$ was prepared on the electronic computer FINAC by the Istituto Nazionale per le Applicazioni del Calcolo, Rome ([2]). I express my sincere thanks to this institution and its director, A. GHIZZETTI for the work done.

*) (Note added on proof, September, 1965.) In the meantime I could find the following partial answer to the question posed on page 92: If $n > 2$, the sequence $\lambda'_n(1), \lambda'_n(1, 2), \dots, \lambda'_n(1, 2, \dots, [n/2])$ consists of strictly increasing numbers.

Indeed denoting $\sum_{p, q=1}^h E_{\min(k_p, k_q)} \left(\frac{2\pi}{n} (p-q) \right)$ by $E_n(k_1, \dots, k_h)$ we have

$$\lambda_n'^2(1, 2, \dots, l-1) = \max_{\substack{k_r=0,1,\dots,n \\ r=1,2,\dots,l-1}} E_n(k_1, \dots, k_{l-1}) = E_n(\hat{k}_1, \dots, \hat{k}_{l-1}),$$

say, and we may write

$$\begin{aligned} & E_n(1, \hat{k}_1, \dots, \hat{k}_{l-1}) + E_n(\hat{k}_1, \hat{k}_2, \dots, \hat{k}_{l-1}, 1) = \\ & = 2E_n(\hat{k}_1, \dots, \hat{k}_{l-1}) + 2 + \sum_{\hat{k}_g > 0} \left\{ \cos \frac{2\pi}{n} (l-g) + \cos \frac{2\pi}{n} g \right\}. \end{aligned}$$

If $l \leq n/2$ then each term of the sum of the right-hand side is non-negative, since $\cos \alpha + \cos \beta \geq 0$ if $\alpha \geq 0, \beta \geq 0, \alpha + \beta \leq \pi$ and so

$$\lambda_n'^2(1, 2, \dots, l-1) < \{E_n(1, \hat{k}_1, \dots, \hat{k}_{l-1}) + E_n(\hat{k}_1, \dots, \hat{k}_{l-1}, 1)\} / 2 \leq$$

$$\leq \max \{E_n(1, \hat{k}_1, \dots, \hat{k}_{l-1}), E_n(\hat{k}_1, \dots, \hat{k}_{l-1}, 1)\}$$

and, a fortiori

$$\lambda_n'^2(1, 2, \dots, l-1) < \lambda_n'^2(1, 2, \dots, l) \quad (l=1, 2, \dots, [n/2])$$

by the definition of $\lambda_n'(1, 2, \dots, l)$.

For the solution of another special case of the problem treated in this paper, see the author's forthcoming article: A property of Dirichlet's kernel (Magyar Tud. Akadémia Mat. Kut. Int. Közl., in the press).

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