

A note on $\mathcal{M}$ -harmonic Besov p -spaces

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Abstract. Two new characterizations of the $\mathcal{M}$ -harmonic Besov space $\mathcal{MB}^p$, $1 < p < \infty$, are given.

1. Introduction

The class of $\mathcal{MB}^p$ -harmonic (and thus holomorphic) Besov p -functions was studied by several authors (see [2], [3], [4], [5], [7], [9], [10], [12]). Motivated by the characterizations of $\mathcal{M}$ -harmonic Bloch space $\mathcal{MB}^\infty$ obtained in [7] we give two new characterizations of the space $\mathcal{MB}^p$, $1 < p < \infty$, (Theorem 1).

The class of $\mathcal{MB}^p$ -functions, $1 < p < \infty$, which contains the space $\mathcal{MB}^p$, was defined and characterized by M. PAVLOVIĆ and the author in [6]. We close this note by stating several additional characterizations of the class $\mathcal{MB}^p$ (Theorem 2 and Theorem 3). Since they are straightforward extensions of the well known results on $\mathcal{M}$ -harmonic and holomorphic Besov p -functions, the proofs will be omitted.

Let B be the open unit ball in $\mathbb{C}^n$ and S the unit sphere in $\mathbb{C}^n$. We denote by ν the normalized Lebesgue measure on B and by σ the rotation invariant probability measure on S .

Let $\tilde{\Delta}$ be the invariant Laplacian on B . That is, $\tilde{\Delta}f(z) = \Delta(f \circ \varphi_z)(0)$, $f \in C^2(B)$, where Δ is the ordinary Laplacian and φ_z the standard automorphism of B taking 0 to z ([11]).

As in [11], we say that a function $f \in C^2(B)$ is $\mathcal{M}$ -harmonic in B , $f \in \mathcal{M}$, if $\tilde{\Delta}f(z) = 0$ for every $z \in B$.

For $f \in C^1(B)$, $Df = \left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n} \right)$, denotes the complex gradient of f , $\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{2n}} \right)$, $z_k = x_{2k-1} + ix_{2k}$, $k = 1, 2, \dots, n$, denotes the real gradient.

For $f \in C^1(B)$ let $\tilde{D}f(z) = D(f \circ \varphi_z)(0)$, $z \in B$, and $\tilde{\nabla}f(z) = \nabla(f \circ \varphi_z)(0)$, $z \in B$, be the invariant complex gradient of f and the invariant real gradient of f respectively.

Let $1 < p < \infty$ and let B^p be the linear space of all functions $f \in C^1(B)$ such that $\|\tilde{\nabla}f\|_{L^p(\tau)} < \infty$, where $d\tau(z) = (1 - |z|^2)^{-n-1}d\nu(z)$. We shall call $\mathcal{M}B^p = \mathcal{M} \cap B^p$, $1 < p < \infty$, $\mathcal{M}$ -harmonic Besov p -space.

Theorem 1. *Let $f \in \mathcal{M}$, $2n < p < \infty$. Then the following statements are equivalent:*

- (i) $f \in \mathcal{M}B^p$,
- (ii) $\int_B (1 - |z|^2)^{p-n-1} |\nabla f(z)|^p d\nu(z) < \infty$,
- (iii) $\int_B (1 - |z|^2)^{p-n-1} (|Rf(z)|^p + |\bar{R}f(z)|^p) d\nu(z) < \infty$.

Here, as usual, $R = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}$, $\bar{R} = \sum_{j=1}^n \bar{z}_j \frac{\partial}{\partial \bar{z}_j}$ are radial derivatives.

2. Proof of Theorem 1

For the proof of Theorem 1 the following lemmas will be needed.

Lemma 2.1 ([7]). *Let $0 < r < 1$. There is a constant C such that if $f \in \mathcal{M}$ then*

- (a) $|T_{ij}Rf(w)| \leq C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |Rf(z)| d\tau(z)$, $w \in B$,
- (b) $|T_{ij}\bar{R}f(w)| \leq C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |\bar{R}f(z)| d\tau(z)$, $w \in B$,
- (c) $|T_{ij}f(w)| \leq C(1 - |w|^2)^{-1/2} \int_{E_r(w)} |f(z)| d\tau(z)$, $w \in B$.

Here, $T_{ij} = \bar{z}_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial z_i}$ denotes the tangential derivative.

For $z \in B$ and r between 0 and 1, $E_r(z) = \{w \in B : |\varphi_z(w)| < r\}$. We set $|E_r(z)| = \nu(E_r(z))$.

In this note we follow the custom of using the letter C to stand for a positive constant which changes its value from one appearance to another while remaining independent of the important variables.

Lemma 2.2. *If $s > 1$, then*

$$\int_0^1 \frac{dt}{|1 - t\langle z, w \rangle|^s} \leq \frac{C}{|1 - \langle z, w \rangle|^{s-1}}, \quad z, w \in B.$$

Lemma 2.3 ([11], p. 17). *If $\alpha > 0$, then*

$$\int_S \frac{d\sigma(\xi)}{|1 - \langle \xi, z \rangle|^{n+\alpha}} = O\left(\frac{1}{(1 - |z|)^\alpha}\right), \quad z \in B.$$

Lemma 2.4. *For $0 < s < t$ we have*

$$\int_0^1 \frac{(1 - \rho)^{s-1}}{(1 - r\rho)^t} d\rho \leq C(1 - r)^{s-t}, \quad 0 \leq r < 1.$$

PROOF of Theorem 1. In [8] it is shown that

$$\begin{aligned} |\tilde{\nabla} f(z)|^2 &= 2(|\tilde{D}f(z)|^2 + |\tilde{D}\bar{f}(z)|^2) \\ (2.1) \quad &= 2(1 - |z|^2)(|Df(z)|^2 - |Rf(z)|^2 + |D\bar{f}(z)|^2 - |R\bar{f}(z)|^2). \end{aligned}$$

An application of Cauchy–Schwarz inequality shows that $|\tilde{\nabla} f(z)| \geq (1 - |z|^2)|\nabla f(z)|$. Thus, (i) $\implies$ (ii).

(ii) $\implies$ (iii). It is easy to see that if (ii) holds then

$$\int_B (1 - |z|^2)^{p-n-1} \left| \frac{\partial f}{\partial z_j}(z) \right|^p d\nu(z) < \infty, \quad 1 \leq j \leq n,$$

and

$$\int_B (1 - |z|^2)^{p-n-1} \left| \frac{\partial f}{\partial \bar{z}_j}(z) \right|^p d\nu(z) < \infty, \quad 1 \leq j \leq n,$$

which in turn implies that

$$\int_B (1 - |z|^2)^{p-n-1} |Rf(z)|^p d\nu(z) < \infty$$

and

$$\int_B (1 - |z|^2)^{p-n-1} |R\bar{f}(z)|^p d\nu(z) < \infty.$$

(iii) $\implies$ (i). Assume, now, that

$$\int_B (1 - |z|^2)^{p-n-1} (|Rf(z)|^p + |R\bar{f}(z)|^p) d\nu(z) < \infty.$$

It is easy to check that

$$|z|^2 |Df(z)|^2 = |Rf(z)|^2 + \sum_{i < j} |T_{ij}f(z)|^2.$$

Using this and (2.1) we find that

$$\begin{aligned} |z|^2 |\tilde{\nabla} f(z)|^2 &= 2(1 - |z|^2) \left[(1 - |z|^2) (|Rf(z)|^2 + |R\bar{f}(z)|^2) \right. \\ &\quad \left. + \sum_{i < j} |T_{ij}f(z)|^2 + \sum_{i < j} |T_{ij}\bar{f}(z)|^2 \right]. \end{aligned}$$

Hence, to show that $f \in \mathcal{MB}^p$ it is sufficient to show that

$$\int_B (1 - |z|^2)^{\frac{p}{2}-n-1} (|T_{ij}f(z)|^p + |T_{ij}\bar{f}(z)|^p) d\nu(z) < \infty, \quad 1 \leq i < j \leq n.$$

An integration by parts show that

$$f(z) = \int_0^1 [Rf(tz) + \bar{R}f(tz) + f(tz)] dt.$$

From this we conclude that it is sufficient to prove that

$$\int_B (1 - |z|^2)^{\frac{p}{2}-n-1} \left(\int_0^1 |T_{ij}u(tz)| dt \right)^p d\nu(z) < \infty, \quad 1 \leq i < j \leq n,$$

where $u(z) = Rf(z)$ or $\bar{R}f(z)$ or $R\bar{f}(z)$ or $\bar{R}\bar{f}(z)$ or $f(z)$.

We will show that, for fixed $1 \leq i < j \leq n$,

$$I = \int_B (1 - |z|^2)^{\frac{p}{2}-n-1} \left(\int_0^1 |T_{ij}Rf(tz)| dt \right)^p d\nu(z) < \infty.$$

The remaining cases may be treated analogously.

Using Lemma 2.1, Fubini's theorem and Lemma 2.2 we find that for any $s > 0$

$$\begin{aligned}
 \int_0^1 |T_{ij} Rf(tz)| dt &\leq C \int_0^1 \left(\int_{E_r(tz)} \frac{|Rf(w)|(1-|w|^2)^s}{|1-\langle tz, w \rangle|^{n+s+\frac{3}{2}}} d\nu(w) \right) dt \\
 &\leq C \int_0^1 \left(\int_B \frac{|Rf(w)|(1-|w|^2)^s d\nu(w)}{|1-t\langle z, w \rangle|^{n+s+\frac{3}{2}}} \right) dt \\
 &= C \int_B |Rf(w)|(1-|w|^2)^s \left(\int_0^1 \frac{dt}{|1-t\langle z, w \rangle|^{n+s+\frac{3}{2}}} \right) d\nu(w) \\
 &\leq C \int_B \frac{|Rf(w)|(1-|w|^2)^s}{|1-\langle z, w \rangle|^{n+s+\frac{1}{2}}} d\nu(w).
 \end{aligned}$$

Applying the continuous form of Minkowski's inequality we obtain

$$\begin{aligned}
 (2.2) \quad I &\leq C \int_0^1 (1-r)^{\frac{p}{2}-n-1} \\
 &\quad \times \left(\int_0^1 \left(\int_S \left(\int_S \frac{|Rf(\rho\xi)|(1-\rho)^s d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi \rangle|^{n+s+\frac{1}{2}}} \right)^p d\sigma(\zeta) \right)^{1/p} d\rho \right)^p dr.
 \end{aligned}$$

By Hölder's inequality

$$\begin{aligned}
 (2.3) \quad &\int_S \frac{|Rf(\rho\xi)| d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi \rangle|^{n+s+\frac{1}{2}}} \\
 &\leq \left(\int_S \frac{|Rf(\rho\xi)| d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi \rangle|^{n+s+\frac{1}{2}}} \right)^{1/p} \left(\int_S \frac{d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi \rangle|^{n+s+\frac{1}{2}}} \right)^{1/p'} \\
 &\leq \frac{C}{(1-r\rho)^{(s+\frac{1}{2})/p'}} \left(\int_S \frac{|Rf(\rho\xi)|^p d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi \rangle|^{n+s+\frac{1}{2}}} \right)^{1/p},
 \end{aligned}$$

by Lemma 2.3. Now we substitute (2.3) into (2.2) and use Fubini's theorem and Lemma 2.3 to get

$$\begin{aligned}
 (2.4) \quad I &\leq C \int_0^1 (1-r)^{\frac{p}{2}-n-1} \left(\int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{(s+\frac{1}{2})/p'}} \right. \\
 &\quad \times \left. \left(\int_S \left(\int_S \frac{|Rf(\rho\xi)|^p d\sigma(\xi)}{|1-\langle r\zeta, \rho\xi \rangle|^{n+s+\frac{1}{2}}} \right) d\sigma(\zeta) \right)^{1/p} d\rho \right)^p dr
 \end{aligned}$$

$$\begin{aligned}
&= C \int_0^1 (1-r)^{\frac{p}{2}-n-1} \left(\int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{(s+\frac{1}{2})/p'}} \right. \\
&\quad \times \left. \left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi) \int_S \frac{d\sigma(\zeta)}{|1-\langle r\zeta, \rho\xi \rangle|^{n+s+\frac{1}{2}}} \right)^{1/p} d\rho \right)^p dr \\
&\leq C \int_0^1 (1-r)^{\frac{p}{2}-n-1} \left(\int_0^1 \frac{(1-\rho)^s}{(1-r\rho)^{s+\frac{1}{2}}} \right. \\
&\quad \times \left. \left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi) \right)^{1/p} d\rho \right)^p dr.
\end{aligned}$$

A simple observation shows that it is possible to select positive parameters s, t_1, t_2, t_3, t_4 such that

- (i) $s = t_1 + t_2 = t_3 + t_4$,
- (ii) $\frac{1}{p'} < t_3 - t_1 < \frac{3}{2} - \frac{n+1}{p}$,
- (iii) $1 - \frac{n+1}{p} < t_2$.

Applying Hölder's inequality on (2.4) and Lemma 2.4 we obtain

$$\begin{aligned}
I &\leq C \int_0^1 (1-r)^{\frac{p}{2}-n-1} \left[\left(\int_0^1 \frac{(1-\rho)^{t_1 p'} d\rho}{(1-r\rho)^{t_3 p'}} \right)^{p/p'} \right. \\
&\quad \times \left. \left(\int_0^1 \frac{(1-\rho)^{t_2 p}}{(1-r\rho)^{(t_4+\frac{1}{2})p}} \left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi) \right) d\rho \right) \right] dr \\
&\leq C \int_0^1 (1-r)^{\frac{3p}{2}-n+(t_1-t_3)p-2} \left(\int_0^1 \frac{(1-\rho)^{t_2 p}}{(1-r\rho)^{(t_4+\frac{1}{2})p}} \right. \\
&\quad \times \left. \left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi) \right) d\rho \right) dr \\
&= C \int_0^1 \left[(1-\rho)^{t_2 p} \left(\int_S |Rf(\rho\xi)|^p d\sigma(\xi) \right) \right. \\
&\quad \times \left. \left(\int_0^1 \frac{(1-r)^{\frac{3p}{2}-n+(t_1-t_3)p-2} dr}{(1-r\rho)^{(t_4+\frac{1}{2})p}} \right) \right] d\rho \\
&= C \int_B (1-|w|^2)^{p-n-1} |Rf(w)|^p d\nu(w) < \infty.
\end{aligned}$$

This finishes the proof of Theorem 1.

3. Class MB^p

For fixed r , $0 < r < 1$, $0 < p \leq \infty$ and $f \in C(B)$, we define

$$\begin{aligned}\widehat{f}(z, r) &= \frac{1}{|E_r(z)|} \int_{E_r(z)} f(w) d\nu(w), \\ MO_p f(z, r) &= \left(\frac{1}{|E_r(z)|} \int_{E_r(z)} |f(w) - \widehat{f}(z, r)|^p d\nu(w) \right)^{1/p}, \\ 0 &< p < \infty, \\ MO_\infty f(z, r) &= \sup \left\{ |f(w) - \widehat{f}(z, r)| : w \in E_r(z) \right\}, \\ MO_p^* f(z, r) &= \left(\frac{1}{|E_r(z)|} \int_{E_r(z)} |f(w) - f(z)|^p d\nu(w) \right)^{1/p}, \\ 0 &< p < \infty,\end{aligned}$$

and

$$MO_\infty^* f(z, r) = \sup \{ |f(w) - f(z)| : w \in E_r(z) \}.$$

A function $f \in C^2(B)$ is said to be of class M_L , $0 < L < \infty$, if $|\widetilde{\Delta} f(z)| \leq Lr^{-2} MO_\infty^* f(z, r)$, for all $z \in B$, $0 < r < 1$, and of class M if $f \in M_L$, for some $L > 0$.

The following characterizations of the M -Besov p -classes $MB^p = M \cap B^p$ are extensions of the well known results on $\mathcal{M}$ -harmonic and holomorphic Besov p -spaces.

Theorem 2. *Let $f \in M$, $1 < q < \infty$, $0 < p \leq q$. Then the following statements are equivalent:*

- (i) $\|Q_p f\|_{L^q(\tau)} < \infty$, where $Q_p f(z) = \|f \circ \varphi_z - f(z)\|_{L^p(\nu)}$, $z \in B$,
- (ii) $f \in MB^q$,

(iii) $\|Qf\|_{L^q(\tau)} < \infty$, where

$$Qf(z) = \sup_{|w|=1} \left\{ \frac{(|\langle Df(z), \bar{w} \rangle|^2 + |\langle D\bar{f}(z), \bar{w} \rangle|^2)^{1/2}}{\sqrt{\langle G_z w, w \rangle}} \right\}$$

and

$$G_z = \frac{1}{2} \left(\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log(1 - |z|^2)^{-n-1} \right).$$

Theorem 3. Let $f \in M$, $\alpha > -1$, $1 < q < \infty$ and $0 < p \leq \infty$. Then the following statements are equivalent:

- (i) $f \in MB^q$,
- (ii) $MO_p^* f(\cdot, r) \in L^q(\tau)$, for all r , $0 < r < 1$,
- (iii) $MO_p^* f(\cdot, r) \in L^q(\tau)$, for some r , $0 < r < 1$,
- (iv) $MO_p f(\cdot, r) \in L^q(\tau)$, for all r , $0 < r < 1$,
- (v) $MO_p f(\cdot, r) \in L^q(\tau)$, for some r , $0 < r < 1$,
- (vi) $\int_B \int_B \frac{|f(z) - f(w)|^q (1 - |z|^2)^\alpha (1 - |w|^2)^\alpha}{|1 - \langle z, w \rangle|^{2n+2+2\alpha}} d\nu(z) d\nu(w) < \infty$.

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