

Some functional equations in connection with a theorem of Dubourdieu

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1. Let us consider four systems of surfaces depending on the parameters x, y, z, u , respectively. In the $u = \text{constant}$ surfaces there is determined a web by the lines $x = \text{constant}$ resp. $y = \text{constant}$ resp. $z = \text{constant}$. Such a web is called a hexagonal one if it is topologically equivalent with a web determined by the lines $x = \text{const.}$ resp. $y = \text{const.}$ resp. $z = \text{const.}$ on the plane $x + y + z = 0$ in the orthogonal coordinate system x, y, z , i. e. the relation

$$(1) \quad U_u^1(x) + U_u^2(y) + U_u^3(z) = 0$$

is satisfied with suitable functions U^i for every fixed $u = \text{const.}$ Similarly, the relations

$$(2) \quad X_x^1(y) + X_x^2(z) + X_x^3(u) = 0,$$

$$(3) \quad Y_y^1(x) + Y_y^2(z) + Y_y^3(u) = 0,$$

$$(4) \quad Z_z^1(x) + Z_z^2(y) + Z_z^3(u) = 0$$

mean that the webs in the $x = \text{const.}$ resp. $y = \text{const.}$ resp. $z = \text{const.}$ surfaces are hexagonal webs. A theorem of Dubourdieu states that under certain differentiability conditions any three of the relations (1)–(4) imply the fourth.

In the present paper we raise some problems in connection with certain functional equations arising in the examination of Dubourdieu's webs.

Problem 1. Which are the solutions U^i, X^i, Y^i, Z^i of the system of functional equations (1)–(4)?

Problem 2. What is the most general form of a ternary operation $u = u(x, y, z)$ satisfying (2)–(4)?

Problem 3. Which are the solutions X^i, Y^i, Z^i of the system of functional equations (2)–(4) for a given $u = u(x, y, z)$?

In the present paper we reduce the problems 2 and 3 to the solution of a system of functional equations containing only three unknown functions of two variables, further, we answer the problems 2 and 3 in the special case where

$$X^1 = X^2 = -X^3, \quad Y^1 = Y^2 = -Y^3, \quad Z^1 = Z^2 = -Z^3.$$

2. Let us consider the functional equations (2)–(4). Suppose that all the functions figuring there are invertible. Without loss of generality, we may suppose that

$$xyz = u(x, y, z)$$

is a loop operation, i. e. there exists a unit element e with the properties:

$$(5) \quad xee = exe = eex = x.$$

In the contrary case we would consider an isotope $I(x, y, z)$ defined by

$$xyz = I(xbc, ayc, abz)$$

with arbitrarily fixed a, b, c . Clearly, this $I(x, y, z)$ is a loop operation with unit element $e = abc$, further, since the hexagonal property is an isotopy invariant one, also $I(x, y, z)$ can be written in the form (2)–(4) with suitable functions X^i, Y^i, Z^i . Thus, in what follows, we assume (5).

With the notation

$$X(t) = -X^3(t)$$

(2) can be written as

$$(6) \quad X(xyz) = X^1(y) + X^2(z).$$

Let us substitute here $z = e$ resp. $y = e$. Then we get

$$X^1(y) = X(xye) - X^2(e), \quad X^2(z) = X(xez) - X^1(e)$$

by which (6) can be written as

$$X(xyz) = X(xye) + X(xez) - X^1(e) - X^2(e).$$

Consequently, with the notation

$$F_x^1(t) = X_x(t) - X_x^1(e) - X_x^2(e)$$

we arrive at

$$(7) \quad F_x^1(xyz) = F_x^1(xye) + F_x^1(xez).$$

Similarly, (3)–(4) imply

$$(8) \quad F_y^2(xyz) = F_y^2(xye) + F_y^2(eyz),$$

$$(9) \quad F_z^3(xyz) = F_z^3(xez) + F_z^3(eyz).$$

By putting here $z = e$ resp. $x = e$, we have by (5) the initial conditions:

$$(10) \quad F_i^i(t) = 0, \quad i = 1, 2, 3.$$

Moreover, by putting $x = e$, $y = e$, and $z = e$ in (7), (8) and (9) respectively, we see that eyz , xez , xye are quasi-additions:

$$(11) \quad \begin{cases} F_e^1(eyz) = F_e^1(y) + F_e^1(z), \\ F_e^2(xez) = F_e^2(x) + F_e^2(z), \\ F_e^3(xye) = F_e^3(x) + F_e^3(y). \end{cases}$$

Let

$$(12) \quad F_t^i(u \nabla_t^i v) = F_t^i(u) + F_t^i(v), \quad i = 1, 2, 3$$

define three systems of binary operations ∇_t^i which depend upon a parameter t . Then, e. g., (7) can be written as

$$\begin{aligned} F_x^1(xyz) &= F_x^1(xye) + F_x^1(xez) = \\ &= F_x^1[(xye) \nabla_x^1(xez)] = F_x^1[(x \nabla_e^3 y) \nabla_x^1(x \nabla_e^2 z)] \end{aligned}$$

i. e.

$$xyz = (x \nabla_e^3 y) \nabla_x^1(x \nabla_e^2 z).$$

We obtain similar formulae from (8)–(9). Thus (7)–(11) can be summed up as

$$\begin{aligned} (13) \quad xyz &= (x \nabla_e^3 y) \nabla_x^1(x \nabla_e^2 z) = \\ &= (x \nabla_e^3 y) \nabla_y^2(y \nabla_e^1 z) = \\ &= (x \nabla_e^2 z) \nabla_z^3(y \nabla_e^1 z), \end{aligned}$$

where the operations ∇_t^i are quasi-additions of the form (12) with unit element t :

$$(10') \quad u \nabla_t^i t = t \nabla_t^i u = u, \quad i = 1, 2, 3.$$

Theorem 1. *Supposed that $X^i(t)$, $Y^i(t)$, $Z^i(t)$ are invertible and (5) is fulfilled, then*

(I) *xyz can be represented in the form (13) by means of the binary group operations ∇_t^i for which (12), (10') hold;*

(II) *the functions X^i , Y^i , Z^i can be given by F^i so that*

$$\begin{aligned} X_x^3(t) &= -X_x(t) = -F_x^1(t) - X_x^1(e) - X_x^2(e), \\ X_x^2(t) &= X(xet) - X_x^1(e) = F_x^1(xet) + X_x^2(e), \\ X_x^1(t) &= X(xte) - X_x^2(e) = F_x^1(xte) + X_x^1(e), \\ Y_y^3(t) &= -F_y^2(t) - Y_y^1(e) - Y_y^2(e), \\ Y_y^2(t) &= F_y^2(eyt) + Y_y^2(e), \\ Y_y^1(t) &= F_y^2(tye) + Y_y^1(e), \\ Z_z^3(t) &= -F_z^3(t) - Z_z^1(e) - Z_z^2(e), \\ Z_z^2(t) &= F_z^3(etz) + Z_z^2(e), \\ Z_z^1(t) &= F_z^3(tez) + Z_z^1(e) \end{aligned}$$

are true, where $X_s^i(e)$, $Y_s^i(e)$, $Z_s^i(e)$ are arbitrary functions of the variables and eyz , xex , xye are arbitrary group operations expressible by means of F_e^i in the form (11).

Thus problems 2 and (3) reduce to the determination of the functions $F_t^i(s)$ satisfying (7)–(11).¹⁾

Remark that the special case

$$u\nabla_t^i v = u\nabla_e^i t^{-1}\nabla_e^i v$$

of (13) was treated previously in [3].

3. Let us consider the special case

$$-X^3 = X^1 = X^2 = X, \quad -Y^3 = Y^1 = Y^2 = Y, \quad -Z^3 = Z^1 = Z^2 = Z.$$

Then (2)–(4) are specialized as

$$(14) \quad xyz = X_x^{-1}[X_x(y) + X_x(z)] = Y_y^{-1}[Y_y(x) + Y_y(z)] = Z_z^{-1}[Z_z(x) + Z_z(y)].$$

From this it follows that

$$xyz = xzy = zyx = yxz.$$

Hence (14) can be simplified as

$$(15) \quad xyz = F_x^{-1}[F_x(y) + F_x(z)] = F_y^{-1}[F_y(x) + F_y(z)] = F_z^{-1}[F_z(x) + F_z(y)].$$

Since $x * y = xty$ is a binary group operation for every fixed t , we have

$$(xty)tz = xt(ytz),$$

i. e.

$$F_z^{-1}(F_z\{F_x^{-1}[F_x(t) + F_x(y)]\} + F_z(t)) = F_x^{-1}(F_x(t) + F_x\{F_z^{-1}[F_z(y) + F_z(t)]\})$$

which, by the new variables

$$u = F_x(t), \quad v = F_x(y)$$

and by the notation

$$(16) \quad f(s) = F_z[F_x^{-1}(s)],$$

can be written as

$$f^{-1}[f(u+v) + f(u)] = u + f^{-1}[f(u) + f(v)].$$

Now we define

$$(17) \quad M(u, v) = f^{-1}[f(u) + f(v)]$$

by which we obtain the functional equation

$$(18) \quad M(u+v, u) = u + M(u, v), \quad M(u, v) = M(v, u).$$

Lemma. *The most general continuous solution of (18) is*

$$(19) \quad M(u, v) = u + v + c,$$

where c is an arbitrary constant.

¹⁾ A similar result can be established also for the more general case where $x+y$ is a group operation but not necessarily abelian ([2]).

PROOF. First we write (18) in the symmetric form

$$(20) \quad M(u, v) + u = M(u, v + u), \quad M(u, v) + v = M(u + v, v).$$

The repeated application of (20) gives

$$(21) \quad M(u, v) + nu = M(u, v + nu), \quad M(u, v) + mv = M(u + mv, v)$$

for every integer $n, m \geq 0$. If we write new variables $v_1 = v + nu, u_1 = u + mv$, then (21) becomes

$$M(u, v_1 - nu) = M(u, v_1) - nu, \quad M(u_1 - mv, v) = M(u_1, v) - mv,$$

hence (21) remains valid for every integer n, m .

Both equations in (21) can be united in the form

$$(22) \quad M[u + mv, v + n(u + mv)] = M(u, v) + mv + n(u + mv), \quad n, m = 0, \pm 1, \pm 2, \dots$$

Let us introduce

$$(23) \quad N(u, v) = M(u, v) - u - v,$$

then (22) becomes

$$(24) \quad N[u + mv, v + n(u + mv)] = N(u, v).$$

Now, choosing incommensurable u and v , by means of suitable integers m_k, n_k we can define a sequence

$$u_1 = u + m_1 v, v_1 = v + n_1 u_1, \dots, u_{k+1} = u_k + m_{k+1} v_k, v_{k+1} = v_k + n_k u_k, \dots$$

such that

$$|u_{k+1}| < |u_k|, \quad |v_{k+1}| < |v_k|.$$

Then, taking (24) into account, we get

$$N(u, v) = N(u_1, v_1) = N(u_k, v_k) = \dots = N\left(\lim_{k \rightarrow \infty} u_k, \lim_{k \rightarrow \infty} v_k\right) = N(0, 0) = c.$$

Thus, by (23), we have proved

$$M(u, v) = u + v + c$$

for any incommensurable u, v . However, the set of pairs of such u, v -s is everywhere dense in the plane (u, v) , therefore our lemma is proved, since $M(u, v)$ is supposed to be continuous.

By our lemma and by (17) we obtain

$$f(u) + f(v) = f(u + v + c),$$

therefore,

$$f(u - c) = F_z[F_x^{-1}(u - c)] = a_{z,x}(u) \quad (c = c(z, x))$$

is an additive function for every fixed z, x . Thus we get

$$F_z(t) = a_{z,x}[F_x(t) + c] = a_{z,x}[F_x(t)] + b_{z,x} = a_{z,0}[F_0(t)] + b_{z,0},$$

$$F_z^{-1}(t) = F_0^{-1}[a_{z,0}^{-1}(t - b_{z,0})], \quad b_{z,x} = a_{z,x}[c(z, x)],$$

by which (15₃) becomes

$$xyz = F_z^{-1}[F_z(x) + F_z(y)] = F_0^{-1}(a_{z,0}^{-1}\{a_{z,0}[F_0(x)] + b_{z,0} + a_{z,0}[F_0(y)] + b_{z,0} - b_{z,0}\}) = F_0^{-1}[F_0(x) + F_0(y) + a_{z,0}^{-1}(b_{z,0})] = F_0^{-1}[F_0(x) + F_0(y) + G(z)],$$

where

$$G(z) = a_{z,0}^{-1}(b_{z,0}) = c(z, 0).$$

Here, because of $xyz = xzy$, i. e.

$$F_0^{-1}[F_0(x) + F_0(y) + G(z)] = F_0^{-1}[F_0(x) + F_0(z) + G(y)],$$

we have

$$F_0(y) + G(z) = F_0(z) + G(y),$$

thus

$$G(z) - F_0(z) = a, \quad G(z) = F_0(z) + a.$$

So xyz can be written as

$$xyz = F_0^{-1}[F_0(x) + F_0(y) + F_0(z) + a] = F^{-1}[F(x) + F(y) + F(z)],$$

where

$$F(t) = F_0(t) + a/2.$$

Theorem 2. Every continuous ternary quasigroup operation xyz satisfying (14) is a quasi-addition:

$$(25) \quad xyz = F^{-1}[F(x) + F(y) + F(z)].$$

4. Finally, let us examine the solution of (14) e. g. for $X_x(t)$ in the special case where (25) holds. Then we must solve the functional equation

$$(26) \quad F^{-1}[F(x) + F(y) + F(z)] = X_x^{-1}[X_x(y) + X_x(z)].$$

With other variables and with the notation

$$G_x(y) = X_x(y'), \quad s' = F^{-1}(s)$$

we get

$$G_x(x + y + z) = G_x(y) + G_x(z).$$

By putting $y = s - x$, $z = t - x$, this shows that

$$G_x(t - x) = a_x(t)$$

is an additive function of t for every fixed x . Taking the definition of G into account, we get

$$X_x(y) = G_{F(x)}[F(y)] = a_{F(x)}[F(x) + F(y)] = a_x^*[F(x) + F(y)].$$

This function $X_x(y)$ with an arbitrary additive $a_x^*(t)$ satisfies (26) as

$$\begin{aligned} X_x(xyz) &= a_x^*[F(x) + F(x) + F(y) + F(z)] = a_x^*[F(x) + F(y)] + a_x^*[F(x) + F(z)] = \\ &= X_x(y) + X_x(z) \end{aligned}$$

holds.

Theorem 3. *The most general invertible solution of the functional equation (26) is*

$$X_x(t) = a_x[F(x) + F(y)],$$

where $a_x(t)$ is an arbitrary invertible and additive function of the variable t .

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(Received August 22, 1964.)