

On the geometric means of integral functions^{*)}

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1. Let $f(z)$ be an integral function of order ϱ . Let $n(r)$ denote the number of zeros of $f(z)$ for $|z| \leq r$ and let $N(r) = \int_0^r \frac{n(x)}{x} dx$, $f(0) \neq 0$. Let $G(r)$ and $g(r)$ denote the geometric means of $|f(z)|$ on the circumference $|z| = r$ and $|z| \leq r$ respectively. Then ([1], pp. 144, 108)

$$(1.1) \quad G(r) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \right\}$$

$$(1.2) \quad g(r) = \exp \left\{ \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \log |f(xe^{i\theta})| x dx d\theta \right\}.$$

Consider a function $\varphi(r)$ with the following properties: (i) $\varphi(r)$ is continuous for $r > r_0$; (ii) $\varphi(r) > 0$ for $r > r_0$ and (iii) $\varphi(lr) \sim \varphi(r)$ as $r \rightarrow \infty$ for every constant $l > 0$. Further, for $0 < \varrho < \infty$, let

$$(1.3) \quad \lim_{r \rightarrow \infty} \sup \frac{\log g(r)}{r^\varrho \varphi(r)} = \begin{cases} p, & (0 < \varrho \leq p < \infty); \\ \lim_{r \rightarrow \infty} \sup \frac{N(r)}{r^\varrho \varphi(r)} = \begin{cases} c, & (0 < d \leq c < \infty). \end{cases} \end{cases}$$

We shall obtain some of the properties of $G(r)$ and $g(r)$.

2. Theorem 1. *If $f(z)$ be an integral function of finite order ϱ*

$$(2.1) \quad \frac{2}{(\varrho+2)} \frac{d}{c} \leq \limsup_{r \rightarrow \infty} \frac{\log g(r)}{N(r)} \leq \frac{2}{(\varrho+2)} \frac{c}{d},$$

where c and d are given by (1.3).

PROOF. From (1.2), we have

$$\log g(r) = \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \log |f(xe^{i\theta})| x dx d\theta = \frac{2}{r^2} \int_0^r \left\{ \int_0^x \frac{n(t)}{t} dt + \log |f(0)| \right\} x dx,$$

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on using Jensen's theorem,

$$\log g(r) = \frac{2}{r^2} \int_0^r N(x)x \, dx + \log |f(0)|.$$

Therefore,

$$(2.2) \quad \frac{1}{2} r^2 \log \left\{ \frac{g(r)}{|f(0)|} \right\} = \int_0^r N(x)x \, dx.$$

Writing (2.2) as

$$\begin{aligned} \frac{1}{2} r^2 (1+\delta)^2 \log \left[\frac{g\{r(1+\delta)\}}{|f(0)|} \right] &= \int_0^{r_0} N(x)x \, dx + \int_{r_0}^r N(x)x \, dx + \int_r^{(1+\delta)r} N(x)x \, dx \\ &< O(1) + (c+\varepsilon) \int_{r_0}^r x^{q+1} \varphi(x) \, dx + N\{r(1+\delta)\} r^2 \delta \left(1 + \frac{\delta}{2} \right) \sim \\ &\sim (c+\varepsilon) \varphi(r) \frac{r^{q+2}}{(q+2)} + N\{r(1+\delta)\} r^2 \delta \left(1 + \frac{\delta}{2} \right), \quad \text{by [2], Lemma 5} \quad \delta \geq 0 \end{aligned}$$

Proceeding to limits and using (1.3), we get

$$(2.3) \quad \frac{1}{2} (1+\delta)^{q+2} p \leq \frac{c}{(q+2)} + c\delta(1+\delta)^q \left(1 + \frac{\delta}{2} \right).$$

Similarly, we obtain

$$(2.4) \quad \frac{1}{2} (1+\delta)^{q+2} q \geq \frac{d}{(q+2)} + d\delta \left(1 + \frac{\delta}{2} \right).$$

Putting $\delta=0$ in (2.3) and (2.4), we have

$$(2.5) \quad \frac{p}{2} \leq \frac{c}{(q+2)} \quad \text{and} \quad \frac{q}{2} \geq \frac{d}{(q+2)}$$

From (1.3), we obtain

$$\frac{q-\varepsilon}{c+\varepsilon} < \frac{\log g(r)}{N(r)} < \frac{p+\varepsilon}{d-\varepsilon}.$$

Taking limits and using (2.5) we get

$$\frac{2}{(q+2)} \frac{d}{c} \leq \lim_{r \rightarrow \infty} \sup \inf \frac{\log g(r)}{N(r)} \leq \frac{2}{(q+2)} \frac{c}{d}.$$

Corollary 1. If $c=d$,

$$(q+2) \log g(r) \sim 2N(r).$$

3. Theorem 2. *If $\log g(r) \sim pr^q \Phi(r)$ then $N(r) \sim \frac{(q+2)}{2} pr^q \varphi(r)$ and conversely.*

PROOF. From (2. 5), if $c=d$, $p=q=\frac{2c}{q+2}$.

Suppose now $p=q$. We shall show that $c=d$. If $0 < \eta < 1$, we have from (2. 2)

$$\begin{aligned} N(r)\eta &< \frac{1}{r^2} \int_r^{r+\eta r} N(x)x \, dx = \frac{1}{2} (1+\eta)^2 \log \left\{ \frac{g(r+\eta r)}{|f(0)|} \right\} - \frac{1}{2} \log \left\{ \frac{g(r)}{|f(0)|} \right\} = \\ &= \frac{1}{2} p(1+\eta)^{q+2} r^q \varphi(r+\eta r) - \frac{1}{2} pr^q \varphi(r) + o(r^q \varphi(r)). \end{aligned}$$

Hence,

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{r^q \varphi(r)} \leq \frac{p}{2} (q+2) + \beta\eta,$$

where β is a constant. Since η is arbitrary we get

$$\limsup_{r \rightarrow \infty} \frac{N(r)}{r^q \varphi(r)} \leq \frac{p}{2} (q+2).$$

By considering the integral $\frac{1}{2} \log \left\{ \frac{g(r)}{|f(0)|} \right\} - \frac{(1-\eta)^2}{2} \log \left\{ \frac{g(r-\eta r)}{|f(0)|} \right\}$, we get

$$\liminf_{r \rightarrow \infty} \frac{N(r)}{r^q \varphi(r)} \geq \frac{p}{2} (q+2)$$

and hence

$$N(r) \sim \frac{p}{2} (q+2) r^q \varphi(r).$$

4. Theorem 3. *If $f(z)$ ($f(0) \neq 0$) is an integral function of order q ,*

$$\limsup_{r \rightarrow \infty} \frac{\log \left\{ r \frac{G'(r)}{G(r)} \right\}}{\log r} \leq q,$$

where $G'(r)$ is the derivative of $G(r)$ and $r \rightarrow \infty$ through almost all values of r .

PROOF. From (1. 1), we have

$$(4. 1) \quad \log G(r) = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta = \int_0^r \frac{n(x)}{x} \, dx + \log |f(0)|,$$

on using Jensen's theorem.

Differentiating (4.1) with respect to r , we get

$$(4.2) \quad \frac{G'(r)}{G(r)} = \frac{n(r)}{r}$$

for almost all values of r .

Taking limits on both the sides of (4.2), we get

$$\limsup_{r \rightarrow \infty} \frac{\log \left\{ r \frac{G'(r)}{G(r)} \right\}}{\log r} = \limsup_{r \rightarrow \infty} \frac{\log n(r)}{\log r} \equiv \varrho,$$

where $r \rightarrow \infty$ through almost all values of r .

Further, we note that if $f(z)$ ($f(0) \neq 0$) is an integral function of finite non-integral order ϱ ,

$$\limsup_{r \rightarrow \infty} \frac{\log \left\{ r \frac{G'(r)}{G(r)} \right\}}{\log r} = \varrho,$$

where $r \rightarrow \infty$ through almost all values of r .

5. Theorem 4. *If $f(z)$ ($f(0) \neq 0$) is an integral function of order ϱ and finite-type T ,*

$$(i) \quad \limsup_{r \rightarrow \infty} \frac{\log G(r)}{r^\varrho \log r} \leq \limsup_{r \rightarrow \infty} \frac{n(r)}{r^\varrho} \leq e\varrho T,$$

$$(ii) \quad \liminf_{r \rightarrow \infty} \frac{\log G(r)}{r^\varrho \log r} \leq \liminf_{r \rightarrow \infty} \frac{n(r)}{r^\varrho} \leq \varrho T.$$

PROOF. From (4.1), we have

$$\log G(r) = \int_{r_0}^r \frac{n(x)}{x} dx + O(1) \leq n(r) (\log r - \log r_0) + O(1).$$

Taking limits and using (2.5.14) and (2.5.15) [3], p. 16, we get (i) and (ii).

References

- [1] G. PÓLYA and G. SZEGŐ, *Aufgaben und Lehrsätze aus der Analysis I*, Berlin, 1925.
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- [3] R. P. BOAS, *Entire Functions*, New York, 1954.

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