

On the number of independent complete subgraphs

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At the Symposium on theory of graphs in Smolenice P. ERDŐS has asked the following problem (see [1]):

Let $l > 1$, assume that G has ln vertices and every vertex has valency $\cong (l-1)n$. Is it then true that G contains n independent complete l -gons (i. e. no two of which have a common vertex).

This conjecture was not still proved in general, but it was proved for some values of l at arbitrary n . For $l=2$ it is a consequence of a well known theorem of Dirac and for $l=3$ it was proved by CORRÁDI and HAJNAL in [2]. For $l=4$ CORRÁDI proved that G contains $n-1$ independent complete quadrilaterals and an n -th complete quadrilateral independent of the previous ones with perhaps one edge missing.

In this paper one goes to the problem from another side. The problem is solved for arbitrary l , but only for the values $n=2$ and $n=3$. (For $n=1$ the conjecture holds trivially.) Let $n=2$. The graph G has $2l$ vertices and every vertex has valency $\cong 2l-2$. So in the complement $\bar{G}$ of G every vertex has valency $\cong 1$, i. e. every component of $\bar{G}$ is either a single edge with its end vertices, or an isolated vertex. Therefore we can number the vertices of $\bar{G}$ as follows. If two vertices are joined by an edge in $\bar{G}$, we assign the number 1 to one of them and the number 2 to the other. Then we decompose the set of isolated vertices of $\bar{G}$ into two disjoint sets of equal cardinality (this is possible, as the number of isolated vertices is even), to all vertices

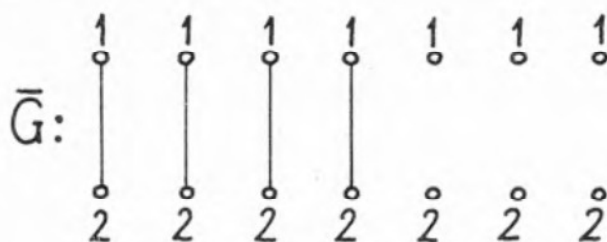


Fig. 1

of one of the sets we assign the number 1, to all vertices of the other set we assign the number 2 (see fig. 1). The set of all vertices numbered by 1 (resp. by 2) consists of l vertices and is evidently independent in $\bar{G}$, therefore it generates a complete l -gon in G .

Let $n=3$. The graph G has $3l$ vertices and every vertex has valency $\equiv 3l-3$. So in the complement $\bar{G}$ of G every vertex has valency $\equiv 2$; i. e. every component of $\bar{G}$ is either a circuit, or a path, or an isolated vertex. Now we shall construct the set K as follows. If a circuit in $\bar{G}$ has the number of vertices congruent with 2 modulo 3, we choose two neighbouring vertices of that circuit and these will be contained in K . If a circuit in $\bar{G}$ has the number of vertices congruent with 1 modulo 3, we choose one vertex of that circuit and it will be contained in K . All vertices of $\bar{G}$ which are not contained in any circuit will be also contained in K . Other vertices are not contained in K . All components of the subgraph of $\bar{G}$ generated by the set K are paths or isolated vertices and the cardinality of K is divisible by three (as the number of vertices not contained in K is evidently divisible by three). So it is possible to number all vertices of K by the numbers 1, 2 and 3, so that neighbouring vertices in $\bar{G}$ might have different numbers and the sets of vertices equally numbered have equal cardinality. When a circuit has the number of vertices congruent with 0 modulo 3, we can go around it, starting with an arbitrary vertex, and number its vertices mutually by 1, 2 and 3. Now let a_1, a_2, a_3 be some permutation of the numbers 1, 2, 3. When a circuit has the number of vertices congruent with 1 modulo 3 and its vertex belonging to K has number a_1 , we start in one vertex neighbouring with it and go along the circuit (with exception of the vertex belonging to K) numbering its vertices mutually by a_2, a_1, a_3 . When a circuit has the number of vertices congruent with 2 modulo 3 and its vertices belonging to K have numbers a_1 and a_2 , we start in one vertex neighbouring with the vertex numbered by a_1 and go along the circuit (with exception of the vertices belonging to K) numbering its vertices

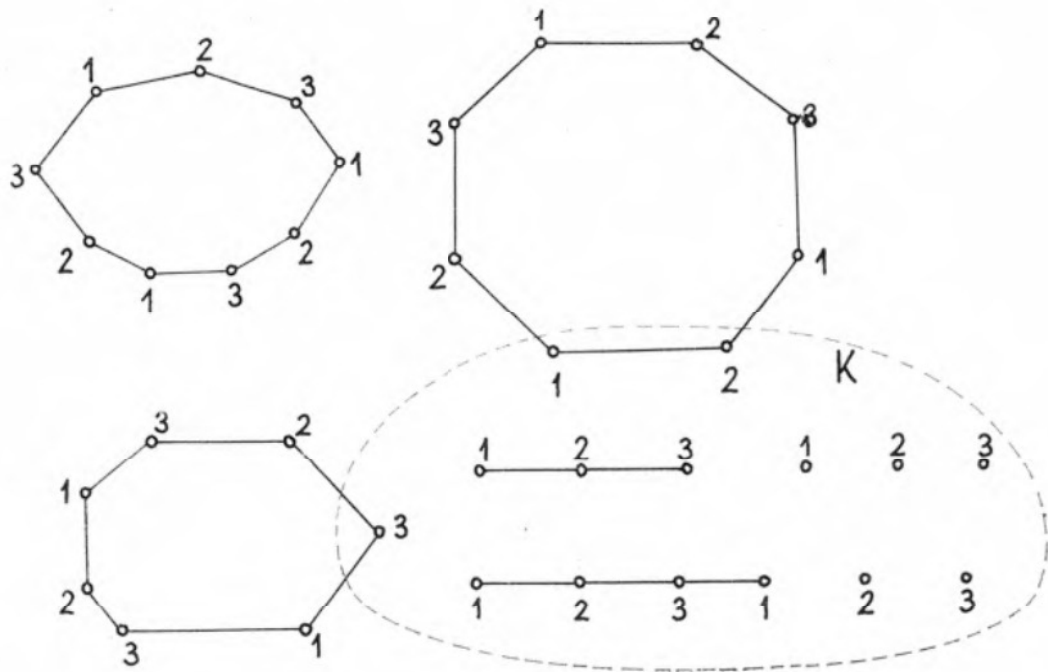


Fig. 2

mutually by a_2, a_3, a_1 . This numbering is seen on fig. 2. In such a numbering the set of vertices numbered by the same number has again the cardinality l and is independent in $\bar{G}$, therefore it generates a complete l -gon in G .

References

- [1] Theory of Graphs and Its Applications. Proceedings of the Symposium held in Smolenice, June 1963. *Praha* (1964).
- [2] K. CORRÁDI—A. HAJNAL, On the number of independent circuits in a graph, *Acta Math. Acad. Sci. Hungar.* **14** (1963), 423—439.

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