

On the equation $\varphi(\sum x_i) = \sum a_{ij} \varphi(x_i) g(x_i)$

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It is known that the functions $\varphi(x) = Bx$ ($B = \text{const.}$) can be characterized as the only solutions having points of continuity of Cauchy's equation

$$\varphi(x+y) = \varphi(x) + \varphi(y).$$

This equation is equivalent to the equation

$$(1) \quad \varphi(x_1 + \dots + x_n) = \sum_{k=1}^n \varphi(x_k)$$

(cf. ACZÉL—KIESEWETTER [3]).

A natural generalization of equation (1) is the equation

$$(2) \quad \varphi(x_1 + \dots + x_n) = \sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij} \varphi(x_i) g(x_j),$$

where φ and g are unknown, a_{ij} constants.

Can equation (2) serve to a characterization of some type of functions? Yes. For $n \geq 3$ it is characteristic for the functions $\varphi(x) = Bx + C$ ($B = \text{const.}$, $C = \text{const.}$).

If $n = 2$, the solutions of equation (2) can be obtained from those of the equation

$$(3) \quad \varphi(x+y) = \varphi(x)g(y) + \varphi(y)g(x)$$

which was considered e. g., in [1], [4], [5], [6].

By the results of [5] it is easy to show that in the case $n = 2$ the only non-zero solutions φ, g of equation (2) which have at least one point of continuity in common are the following couples of functions

$$1^\circ \quad \varphi(x) = Ae^{ax}, \quad g(x) = \frac{1}{a_{12} + a_{21}} e^{ax}$$

$$2^\circ \quad \varphi(x) = Axe^{ax}, \quad g(x) = \alpha e^{ax},$$

$$3^\circ \quad \varphi(x) = Ae^{ax} \sin bx, \quad g(x) = \alpha e^{ax} \cos bx,$$

$$4^\circ \quad \varphi(x) = Ae^{ax} \sinh bx, \quad g(x) = \alpha e^{ax} \cosh bx,$$

where A, a, b ($A \neq 0, b \neq 0$) are arbitrary constants, and $\alpha = \frac{1}{a_{12}} = \frac{1}{a_{21}}$.

It can be proved that for $n \geq 3$ the only non-zero solutions of equation (2) which have a point of continuity in common are

$$1^\circ \quad \varphi(x) = Bx + C, \quad g(x) = \frac{1}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}},$$

$$2^\circ \quad \varphi(x) = Bx, \quad g(x) = \frac{n}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}} = \frac{1}{\sum_{\substack{j=1 \\ i \neq j}}^n a_{ij}},$$

where B, C ($C \neq 0, B \neq 0$ in 2°) are arbitrary constants.

Thus the functions $\varphi(x) = Bx + C$ can be defined as the only functions having at least one point of continuity for which there exists a function g continuous at this point, such that φ, g constitute the solution of equation (2).

Before we solve equation (2), we shall prove some theorems on the regularity of its solutions. We shall restrict ourselves to the case $n \geq 3$, since for $n = 2$ equation (2) can easily be reduced to equation (3) for which adequate theorems were given in [5]. (Notice that the proof of Theorem I remains valid for $n = 2$, too).

Theorem 1. *If the functions φ, g which satisfy equation (2) have at least one point of continuity in common and $\varphi(x) \neq 0$, they are continuous everywhere.*

PROOF. Let a be a common point of continuity of the functions φ and g .

In order to prove that the function φ is continuous at an arbitrary point ξ , we fix $x_1, \dots, x_{n-1}$ such that $x_1 + \dots + x_{n-1} = \xi - a$. If $x_n \rightarrow a, x_1 + \dots + x_n \rightarrow \xi$. Since the right side of equation (2) is continuous at a , the function φ is continuous at ξ .

We shall show that the function g too is continuous everywhere. Since $\varphi(x) \neq 0$, it follows from (2) that $\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij} \neq 0$ and for some j it must be $\sum_{\substack{i=1 \\ i \neq j}}^n a_{ij} \neq 0$. Without loss

of generality we can assume that $\sum_{i=1}^{n-1} a_{in} \neq 0$. Given b such that $\varphi(b) \neq 0$ we take $x_1 = \dots = x_{n-1} = b, x_n = x$. Then (2) can be written as

$$\varphi((n-1)b + x) = \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} a_{ij} \varphi(b)g(b) + \sum_{i=1}^{n-1} a_{in} \varphi(b)g(x) + \sum_{j=1}^{n-1} a_{nj} \varphi(x)g(b)$$

whence

$$(4) \quad g(x) = \frac{\varphi((n-1)b + x) - \sum_{j=1}^{n-1} a_{nj} g(b) \varphi(x) - \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} a_{ij} \varphi(b) g(b)}{\sum_{i=1}^{n-1} a_{in} \varphi(b)}$$

and it follows from the continuity of the function φ that the function g is continuous everywhere.

Now we can make use of Aczél's method (cf. [2]) and prove the following

Theorem 2. *If the functions φ, g satisfy equation (2), are continuous, and moreover $\varphi(x) \not\equiv 0$, they are functions of class C^∞ .*

PROOF. Consider first the case $\varphi(0) \neq 0$, e. g. $\varphi(0) = 1$. Then

$$g(0) = \frac{1}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}},$$

Similarly as in the proof of Theorem 1 we can assume that $\sum_{i=1}^{n-1} a_{in} \neq 0$. Substituting $x_1 = \dots = x_{n-1} = 0, x_n = x$ into (2) we obtain

$$\varphi(x) = \sum_{j=1}^{n-1} a_{nj} \varphi(x) g(0) + \sum_{i=1}^{n-1} a_{in} g(x) + \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} a_{ij} g(0).$$

Hence

(5)
where

$$g(x) = A\varphi(x) + B,$$

$$A = \left(1 - \frac{\sum_{j=1}^{n-1} a_{nj}}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}} \right) \frac{1}{\sum_{i=1}^{n-1} a_{in}},$$

$$B = - \frac{\sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} a_{ij}}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij} \sum_{i=1}^{n-1} a_{in}}.$$

Substituting (5) into (2) we obtain

$$\varphi(x_1 + \dots + x_n) = \sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij} \varphi(x_i) (A\varphi(x_j) + B).$$

Putting $x_1 = \dots = x_{n-1} = s, x_n = x$ and integrating by s from α to β , we obtain

$$(6) \quad \int_{\alpha}^{\beta} \varphi((n-1)s + x) ds = C\varphi(x) + D,$$

where

$$C = A \sum_{i=1}^{n-1} (a_{in} + a_{ni}) \int_{\alpha}^{\beta} \varphi(s) ds + B \sum_{j=1}^{n-1} a_{nj} (\beta - \alpha),$$

$$D = \sum_{\substack{i,j=1 \\ i \neq j}}^{n-1} a_{ij} \int_{\alpha}^{\beta} \varphi(s) (A\varphi(s) + B) ds + B \sum_{i=1}^{n-1} a_{in} \int_{\alpha}^{\beta} \varphi(s) ds.$$

If $\varphi(x) \neq \text{const}$, one can choose the numbers α, β such that $C \neq 0$. Then we obtain from (6)

$$\varphi(x) = \frac{1}{C} \int_{\alpha}^{\beta} \varphi((n-1)s+x) ds - \frac{D}{C}.$$

Since the function φ is continuous, the right hand side of the last equality is differentiable, hence it follows that the function φ is differentiable.

Now, we can differentiate the right hand side of equation (2) with respect to $x_1, \dots, x_n$, and therefore the function φ which appears on the left hand side of equation (2) must have derivatives of order $\leq n$. From (4) it follows that this is the case also for the function g .

Similarly we conclude that the derivatives of order $> n$ exist.

If $\varphi(x) = \text{const}$, the theorem is trivial (in view of (4)).

In the case $\varphi(0) = 0$ we substitute $x_i = x, x_j = 0$ for $j = 1, \dots, n, j \neq i$ and we obtain

$$\varphi(x) = \sum_{\substack{j=1 \\ i \neq j}}^n a_{ij} \varphi(x) g(0), \quad (i = 1, \dots, n).$$

Hence

$$(7) \quad g(0) = \frac{1}{\sum_{\substack{j=1 \\ i \neq j}}^n a_{ij}}, \quad (i = 1, \dots, n).$$

Putting $x_k = 0$ for $k = 1, \dots, n, k \neq i, k \neq j$ into (2), we obtain after computations

$$a_{ij} \varphi(x_i) g(x_j) + a_{ji} \varphi(x_j) g(x_i) = \varphi(x_i + x_j) - \sum_{\substack{k=1 \\ k \neq i, k \neq j}}^n (a_{ik} \varphi(x_i) + a_{jk} \varphi(x_j)) g(0).$$

In view of the last equality and in view of (7) we obtain from (2) the equality

$$(8) \quad \varphi(x_1 + \dots + x_n) = \sum_{\substack{i,j=1 \\ i < j}}^n \varphi(x_i + x_j) - (n-2) \sum_{k=1}^n \varphi(x_k).$$

Setting in (8) $x_1 = \dots = x_{n-1} = s, x_n = x$ and integrating for s from α to β , we obtain after computations

$$(9) \quad \varphi(x) = P \int_{\alpha}^{\beta} \varphi((n-1)s+x) ds + Q \int_{\alpha}^{\beta} \varphi(s+x) ds + R,$$

where

$$P = \frac{1}{(n-2)(\beta-\alpha)}, \quad Q = -\frac{n-1}{(n-2)(\beta-\alpha)},$$

$$R = \frac{n-1}{\beta-\alpha} \left[\int_{\alpha}^{\beta} \varphi(s) ds - \frac{1}{2} \int_{\alpha}^{\beta} \varphi(2s) ds \right].$$

Since we can differentiate the right hand side of equality (9), the function φ must have a first derivative. In view of equality (4) (which was obtained in the proof of Theorem 1), we conclude that the first derivative of the function g exists too.

Making use of equation (2) it is easy to show that the second derivative of the function φ also exists. The existence of the second derivative of the function g follows automatically from (4).

The proof of the existence of the higher derivatives of these functions is analogous.

As a simple conclusion from theorem 1 and 2 we get the following

Remark. If the functions φ, g which satisfy equation (2) have at least one point of continuity in common and $\varphi(x) \not\equiv 0$, they are functions of class C^∞ .

Now, one can easily find the solutions φ, g of equation (2) which have a point of continuity in common.

Theorem 3. *The solutions φ, g of equation (2) with $n \geq 3$ which have a point of continuity in common, and such that $\varphi(x) \not\equiv 0$, must have the form*

$$1^\circ \quad \varphi(x) = Bx + C, \quad g(x) = \frac{1}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}},$$

or

$$2^\circ \quad \varphi(x) = Bx, \quad g(x) = \frac{1}{\sum_{\substack{j=1 \\ i \neq j}}^n a_{ij}}, \quad (i=1, \dots, n),$$

where B, C ($C \neq 0, B \neq 0$ in 2° are arbitrary constants.

PROOF. By the Remark the functions φ and g have all the derivatives. Differentiating equation (2) for three different variables, we obtain $\varphi'''(x) \equiv 0$. Hence it follows that the function φ has the form

$$(10) \quad \varphi(x) = Ax^2 + Bx + C.$$

Differentiating equation (2) by x_i and x_j ($i \neq j$), we obtain

$$\varphi''(x_1 + \dots + x_n) = a_{ij} \varphi'(x_i) g'(x_j) + a_{ji} \varphi'(x_j) g'(x_i).$$

Hence, in view of (10), we have

$$(11) \quad 2A = a_{ij}(2Ax_i + B)g'(x_j) + a_{ji}(2Ax_j + B)g'(x_i).$$

Differentiating (11) by x_i and x_j ($i \neq j$) and putting $x_i = x_j = x$, we obtain

$$(a_{ij} + a_{ji})Ag''(x) = 0.$$

It must be $Ag''(x) = 0$, since in the contrary case we would have $a_{ij} + a_{ji} = 0$ for $i, j = 1, \dots, n, i \neq j$, and then (in view of (2)) $\varphi(x) \equiv 0$.

Now, we shall prove that $g''(x) \equiv 0$. In fact, $Ag''(x) = 0$ implies either $g''(x) \equiv 0$ or $A = 0$. If $A = 0$, it follows from (11) that $(a_{ij} + a_{ji})Bg'(x) = 0$ ($i, j = 1, \dots, n, i \neq j$). Since the assumption $Bg'(x) \neq 0$ yields $\varphi(x) \equiv 0$, it must be either $B = 0$ or $g'(x) \equiv 0$. We shall prove that $g'(x) \equiv 0$. Indeed, if $B = 0$, then $\varphi(x) = C$ and it follows from (2) that

$$g(x) = \frac{1}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}}.$$

and we have $g'(x) \equiv 0$. Thus it must be $g''(x) \equiv 0$ and $g(x) = Dx + E$.

Equality (11) can be written now as

$$(12) \quad 2A = a_{ij}(2Ax_i + B)D + a_{ji}(2Ax_j + B)D$$

whence we obtain for $x_i = x_j = 0$

$$(13) \quad 2A = (a_{ij} + a_{ji})BD.$$

Differentiating (12) for x_i and x_j ($i \neq j$), we obtain $a_{ij}AD = 0$ and $a_{ji}AD = 0$. It must be $AD = 0$, since in the contrary case we would have $\varphi(x) \equiv 0$. We shall show that $A = 0$. In fact, if $A \neq 0$, then $D = 0$, but this contradicts (13). Thus the function φ has the form

$$\varphi(x) = Bx + C.$$

Now, one can write (12) as $(a_{ij} + a_{ji})BD = 0$. It must be $BD = 0$, since $BD \neq 0$ yields $\varphi(x) \equiv 0$.

If $B = 0$, then $\varphi(x) = C$ ($C \neq 0$) and we conclude from equation (2) that

$$g(x) = \frac{1}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}}.$$

If $D = 0$, then $g(x) = E$ and it results from (2) that

$$E = \frac{1}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}}, \quad \text{for } C \neq 0$$

and

$$E = \frac{n}{\sum_{\substack{i,j=1 \\ i \neq j}}^n a_{ij}} = \frac{1}{\sum_{\substack{j=1 \\ i \neq j}}^n a_{ij}} \quad (i = 1, \dots, n) \quad \text{for } C = 0.$$

Thus the solutions of equation (2) have either form 1° or 2° .

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(Received June 1, 1965.)