

The idempotent separating congruences of a bisimple inverse semigroup with identity *)

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A congruence ρ on a semigroup is called idempotent separating if each congruence class contains at most one idempotent. These congruences have been studied on regular semigroups by HOWIE [3] and MUNN [4]. The main purpose of this note is to determine the idempotent separating congruences of a bisimple inverse semigroup with identity S in terms of certain normal subgroups of the group of units of S . We show that these congruences are uniquely determined by the congruence class containing the identity. We also give a class of examples of bisimple inverse semigroups with identity on which $\mathcal{H}$ (Green's relation) is not a congruence, i. e. $\mathcal{H}$ properly includes each idempotent separating congruence of S .

We will use the terminology and definitions of [2]. Let $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{D}$ be Green's relations [2]. Bisimple inverse Semigroups with identity have been investigated by CLIFFORD [1] and WARNE [7]—[9].

CLIFFORD [1] determined the structure of bisimple inverse semigroups with identity S in terms of their right unit subsemigroups P . Let us briefly review his construction. If $a, b \in P$, there exists $c \in P$ such that $Pa \cap Pb = Pc$. Choose a representative element from each $\mathcal{L}$ -class of P , and let avb be the representative of the $\mathcal{L}$ -class containing c . Define a binary operation $*$ on P by

$$(1) \quad (a * b)b = avb$$

Then, $S \cong P \times P$ under the following definitions of equality and multiplication.

$$(2) \quad (a, b) = (c, d) \text{ if } a = uc, b = ud \text{ where } u \in U, \text{ the group of units of } P.$$

$$(3) \quad (a, b)(c, d) = ((c * b)a, (b * c)d).$$

Conversely, if P is a right cancellative semigroup with identity such that the intersection of two principal left ideals is a principal left ideal, $S = P \times P$ under the above equality and multiplication is a bisimple inverse semigroup with identity with right unit subsemigroup P . The following are also established:

$$(4) \quad a * a \in U$$

$$(5) \quad E_S = ((a, a) : a \in P),$$

*) A summary of a portion of this article appeared as a research announcement in the *Bull. Amer. Math. Soc.* **72** (1966), 679—682.

where E_s is the set of idempotents of S .

(6) (b, a) is the inverse of (a, b) (This is actually established in [7]).

(7) $Pa = Pb$ if and only if $a = ub$ for some $u \in U$.

(8) $((1, u): u \in U) \cong U$ is the group of units of S .

(9) $1 * a = u \in U$ and $a * 1 = ua$.

The following is given in [8]

(10) $(a, b)\mathcal{R}(c, d)$ if $Pa = Pc$

$(a, b)\mathcal{L}(c, d)$ if $Pb = Pd$.

Let us now review the work of Preston [5] on the congruences of an arbitrary inverse semigroup. Let S be an inverse semigroup and let E_S be divided into disjoint sets E_α , where α ranges over some set J , such that $E = \bigcup_{\alpha \in J} E_\alpha$. Suppose further that to each pair $\alpha, \beta \in J$, there corresponds $\gamma \in J$, such that $E_\alpha E_\beta \subseteq E_\gamma$ and that to each $a \in S$ and $\alpha \in J$, there corresponds $\beta \in J$, such that $aE_\alpha a^{-1} \subseteq E_\beta$. For each $\alpha \in J$, let $N(E_\alpha)$ denote an inverse subsemigroup of S with E_α as its set of idempotents. If the $N(E_\alpha)$ satisfy the further conditions:

(A) $aa^{-1}, bb^{-1} \in E_\alpha, a, ab^{-1} \in N(E_\alpha)$, together imply that $b \in N(E_\alpha)$;

(B) $aa^{-1}, bb^{-1} \in E_\alpha, ab^{-1} \in N(E_\alpha)$, together imply that

$$aN(E_\beta)b^{-1} \subseteq N(E_\gamma), \text{ where } aE_\beta a^{-1} \subseteq E_\gamma,$$

then the inverse subsemigroup $N = \bigcup_{\alpha \in J} N(E_\alpha)$, will be said to be a normal subsemigroup of S , with components $N(E_\alpha)$.

If S is an inverse semigroup and ϱ is a congruence on S , the *kernel* of ϱ is the inverse image of $E_{S/\varrho}$ under the canonical homomorphism.

Theorem 1. (PRESTON) Let N , with components $N(E_\alpha)$, be a normal subsemigroup of the inverse semigroup S , and define the relation ϱ_N over S , by $a\varrho_N b$, if and only if, for some α , $aa^{-1}, bb^{-1} \in E_\alpha$ and $ab^{-1} \in N(E_\alpha)$. Then ϱ_N is a congruence relation over S , with kernel N .

Conversely, every congruence relation ϱ over S , has a kernel N which is a normal subsemigroup of S , such that ϱ_N is ϱ . S/ϱ is itself an inverse semigroup with the components of N as its idempotents. A congruence ϱ_N is idempotent separating if and only if $N = U(N_e: e \in E)$ where each N_e is a group and $N_e N_f \subseteq N_{ef}$ and $aN_f a^{-1} \subseteq N_g$ where $g = afa^{-1}$.

Let P be a right cancellative semigroup with identity and group of units U . A subgroup V is called a right normal divisor of P if and only if $aV \subseteq Va$ for all $a \in P$.

The following lemma is established in [8].

Lemma. If S is a bisimple inverse semigroup with identity with right unit subsemigroup P , U , the group of units of P , is a right normal divisor if and only if $\mathcal{H}$ is a congruence on S .

Theorem 2. *Let S be a bisimple inverse semigroup with identity and right unit subsemigroup P . There exists a 1—1 correspondence between the congruences on S such that the congruence class containing the identity is a group and the right normal divisors of P . In this case, each congruence is uniquely determined by the congruence class containing the identity. These congruences are precisely the idempotent separating congruences of S . If ϱ^V is the congruence corresponding to the right normal divisor V , $\varrho_{(a,b)}^V = \{(a, vb) : v \in V\}$. If V_1, V_2 are right normal divisors of P , $V_1 \subseteq V_2$ if and only if $\varrho^{V_1} \subseteq \varrho^{V_2}$. Each idempotent separating congruence of S is contained in $\mathcal{H}$. If $M = \{g \in U \mid xg \in Ux \text{ for all } x \in P\}$, ϱ^M is the maximal idempotent separating congruence of S .*

PROOF. Let V be a right normal divisor of P and let $N_{(a,a)} = \{(a, va) : v \in V\}$. We will show that $N = U(N_{(a,a)} : a \in P)$ is a normal subsemigroup corresponding to the idempotent separating congruence ϱ_N . If $u, v \in V$, by (2), (3), and (4), $(a, ua)(a, va) = (a, uva)$ and it easily follows that $N_{(a,a)}$ is a group. If $u, v_1 \in V$, $a, b \in P$, there exists $s, t \in V$ such that $(b * a)u^{-1} = s(b * a)$ and $(a * b)v_1 = t(a * b)$ since V is a right normal divisor of P . Thus, by (1), (2), (3) and (4), $(a, ua)(b, v_1 b) = (u^{-1}a, a)(b, v_1 b) = ((b * a)u^{-1}a, (a * b)v_1 b) = (s(b * a)a, t(a * b)b) = (avb, s^{-1}t(avb))$ and $N_{(a,a)}N_{(b,b)} \subseteq N_{(avb, avb)}$. Next, let $(a, b) \in S$ and by (5) $(c, c) \in E_S$. Thus, by (3) and (1), $(a, b)(c, c)(b, a) = ((c * b)a, bvc)(b, a) = ((c * b)a, (c * b)a)$. Noting that if $v' \in V$ there exists $w \in V$ such that $(b * c)v' = w(b * c)$, similarly we have $(a, b)(c, v'c)(b, a) = ((c * b)a, (b * c)v'c)(b, a) = ((c * b)a, w(bvc))(b, a) = (w^{-1}(c * b)a, bvc)(b, a) = (w^{-1}(c * b)a, (c * b)a) = ((c * b)a, w(c * b)a)$.

Hence, $(a, b)N_{(c,c)}(b, a) \subseteq N_{((c * b)a, (c * b)a)}$ and the desired result follows from (6) and theorem 1. Conversely, suppose that ϱ is any congruence such that the congruence class containing the identity is a group. Thus, by theorem 1, $\varrho = \varrho_N$ where N is a normal subsemigroup of S . Denote the congruence class containing the identity by N_1 . Suppose that (a, a) and (b, b) are in the same component of N . Thus, by (4) and (2), $(1, a)(a, a)(a, 1) = (1, 1)$ and $(1, a)(b, b)(a, 1) = (1, a)(b, 1)(1, b) \cdot (a, 1) = (b * a, a * b)(a * b, b * a) = (b * a, b * a)$. Thus by theorem 1, $(b * a, b * a) \in N_1$. Thus, $b * a \in U$ by (2). Interchanging a and b , $a * b \in U$. Hence, since $(b * a)a = avb = (a * b)b$ by (1), $(a, a) = (b, b)$ by (2) and ϱ is idempotent separating. By (8), $N_1 = \{(1, v) : v \in V\}$ where V is a subgroup of U . By (9) and theorem 1 if $a \in P$ and $s \in V$, there exists $t \in U$ and $v \in V$ such that $(1, a)(1, s)(a, 1) = (1 * a, (a * 1)s) \cdot (a, 1) = (t, tas)(a, 1) = (1, as)(a, 1) = (a * as, as * a) = (1, v)$. Hence, by (2), there exists $w \in U$ such that $a * as = w$ and $as * a = wv$. Thus, by (1), $was = wva$ and $as = va$, i. e. $aV \subseteq Va$ and V is a right normal divisor of P . Let $N_{(a,a)}$ denote the component of N containing (a, a) . By theorem 1, (4) and (6), $(1, a)N_{(a,a)}(a, 1) \subseteq N_{(1,1)}$. Since $N_{(a,a)}$ is a group, $N_{(a,a)} \subseteq H_{(a,a)}$, the $\mathcal{H}$ -class containing (a, a) . Thus, it follows from (10), (2), and (3) that $N_{(a,a)} = \{(a, wa) : w \in W\}$ where W is a subgroup of U . Now $(1, a)(a, wa)(a, 1) = (1, wa)(a, 1) = (w^{-1}, 1) = (1, w)$ and $w \in V$. Thus, $N_{(a,a)} = \{(a, va) : v \in V\}$. Hence, we have the desired correspondence. Since $(a, ua) = (a, 1)(1, u)(1, a)$ for $u \in U$ by (4) and (2), each component is uniquely determined by the congruence class containing the identity. Next let us show that $\varrho^V \subseteq \mathcal{H}$.

If $(a, b)\varrho^V(c, d)$, $(a, a) = (c, c)$ by (6) and Theorem 1, and $Pa = Pc$ by (2) and (7). Now, again by Theorem 1, $(a, b)(d, c) = ((d * b)a, (b * d)c)$, $(d * b)a = ua$, and $(b * d)c = vc$ for some $u, v \in U$. Thus, $d * b, b * d \in U$ since P is right cancellative. Hence $(d * b)b = bvd = (b * d)d$ and $Pb = Pd$ by (2) and (7). Hence, $(a, b)\mathcal{H}(c, d)$

by (10). We now show that $\varrho_{(c,d)}^V = ((c, ud): u \in V)$. Since $\varrho^V \subseteq \mathcal{H}$, $(a, b) \in \varrho_{(c,d)}^V$ implies that $a = sc$ and $b = td$ where $s, t \in U$ by (7) and (10). Thus, by (4), (2), and theorem 1 there exists $v \in V$ such that $(a, b)(d, c) = (a, b)(t^{-1}b, c) = (a, b)(b, tc) = (a, tc) = (sc, tc) = (c, s^{-1}tc) = (c, vc)$. Hence, since P is right cancellative, using (2), we obtain $s^{-1}t = v \in V$, i. e. $(a, b) = (sc, td) = (c, s^{-1}td)$ and $\varrho_{(c,d)}^V \subseteq ((c, vd): v \in V)$. If $v \in V$, by (4) and (2), $(c, vd)(vd, c) = (c, c)$, $(c, d)(d, c) = (c, c)$, $(c, vd)(d, c) = (v^{-1}c, c) = (c, vc)$ and $(c, vd) \in \varrho^V(c, d)$ by theorem 1. Thus, $\varrho_{(c,d)}^V = ((c, vd): v \in V)$. We next show that $V_1 \subseteq V_2$ if and only if $\varrho^{V_1} \subseteq \varrho^{V_2}$. Suppose that $\varrho^{V_1} \subseteq \varrho^{V_2}$. If $v \in V_1$, $(1, v)\varrho^{V_2} (1, 1)$ and $v \in V_2$, i. e. $V_1 \subseteq V_2$. Clearly, if $V_1 \subseteq V_2$, $\varrho^{V_1} \subseteq \varrho^{V_2}$. Since $M = \{g \in U/xg \in Ux \text{ for all } x \in P\}$ is the greatest right normal divisor of P [6, lemma 2.11], ϱ^M is the maximal idempotent separating congruence of S , q.e.d. By virtue of theorem 2,

if $\mathcal{H}$ is a congruence on S , then $\mathcal{H}$ is the maximal idempotent separating congruence on S . This situation occurs in many cases [8].

However let us now give a class of examples of a bisimple inverse semigroup S with identity on which $\mathcal{H}$ is not a congruence. By the lemma, it is only necessary to construct a right cancellative semigroup P with identity for which U is not a right normal divisor and for which the intersection of two principal left ideals is a principal left ideal and then apply Clifford's structure theorem described above. Let F be the positive part of any ordered field and let $F^* = F \setminus 0$. The required example is given by $P = F^* \times F$ under the multiplication

$$(a, b)(c, d) = (ac, bc + d).$$

It is seen by straightforward calculations that P has the required properties.

Remark. The fact (theorem 2) that each idempotent separating congruence is contained in $\mathcal{H}$ also follows from [4], p. 389, Theorem 2.

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(Received September 27, 1965.)