

On the summability factors of infinite series

By S. M. MAZHAR (Aligarh)

1. Let $\{p_n\}$ be a sequence of non-negative real constants such that $P_n = \sum_{v=0}^n p_v$ tends to infinity with n . The sequence

$$t_n = \frac{1}{P_n} \sum_{v=0}^n p_v s_v,$$

where s_n is the n -th partial sum of a given infinite series $\sum a_n$, defines the $(\bar{N}, p_n)$ means of $\{s_n\}$.

The series $\sum a_n$ is said to be summable $|\bar{N}, p_n|$, if the sequence $\{t_n\}$ is of bounded variation.

If $p_n = \frac{1}{n+1}$, we have $P_n \sim \log n$ and $P_n/\log n \in BV$, and therefore summability $|\bar{N}, \frac{1}{n+1}|$ is equivalent to the summability $|R, \log n, 1|$.

2. Recently BHATT ([1]) has proved the following theorem.

Theorem A. If $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n/n < \infty$ and the sequence $\{s_n\}$ is bounded, then the series $\sum \lambda_n a_n \log n$ is summable $|R, \log n, 1|$.

Later on he [2] gave a generalisation of this theorem in the following form.

Theorem B. If $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n/n < \infty$, and if the sequence $\{k_n\}$, the $(R, \log n, 1)$ mean of the sequence $\{n a_n \log(n+1)\}$, satisfies the condition

$$(2.1) \quad |k_n| = O\{\log(n+1)\}^k (C, 1), \quad k \geq 0,$$

then the series $\sum \lambda_n a_n \{\log(n+1)\}^{1-k}$ is summable $|R, \log n, 1|$.

The object of this paper is to obtain a further generalisation of this theorem.

3. We shall prove the following theorem.

Theorem 1. If $\{\lambda_n\}$ is a convex sequence such that $\sum \lambda_n p_n < \infty$, where $\{p_n\}$ is a non-increasing positive sequence, and the sequence $\{\mu_n\}$, the $(\bar{N}, p_n)$ mean of $\{a_n P_n/p_n\}$, satisfies the condition

$$(3.1) \quad |\mu_n| = O\{\gamma_n\} (C, 1),$$

γ_n being a positive non-decreasing sequence such that $\left| \Delta^2 \frac{1}{\gamma_n} \right| \geq 0$ and $\Delta \gamma_n = O\{p_n \gamma_n / P_n\}$, then the series $\sum a_n \lambda_n P_n / \gamma_n$ is summable $|\bar{N}, p_n|$.

It is clear that if we take $p_n = \frac{1}{n+1}$ and $\gamma_n = \{\log(n+1)\}^k$, Theorem B from our theorem.

4. The following lemmas are pertinent for the proof of this theorem.

Lemma 1. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n p_n < \infty$, where $\{p_n\}$ is a sequence of real positive constants such that $P_n \rightarrow \infty$, then $\{\lambda_n\}$ is a non-negative monotonic decreasing sequence tending to zero and $\lambda_n P_n = o(1)$, as $n \rightarrow \infty$.*

This generalises the following lemma of CHOW ([3]).

Lemma A. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n/n < \infty$, then $\{\lambda_n\}$ is a non-negative decreasing sequence and $\lambda_n \log n = o(1)$, as $n \rightarrow \infty$.*

PROOF OF LEMMA 1. Since $\Delta^2 \lambda_n \geq 0$, it follows that $\Delta \lambda_n$ is non-increasing and λ_n either tends to a finite limit l or to $+\infty$ or $-\infty$. Also since $\Sigma \lambda_n p_n < \infty$ we have

$$(4.1) \quad \frac{1}{P_n} \sum_1^n \lambda_m p_m = o(1), \quad (n \rightarrow \infty).$$

Now let $\lim_{n \rightarrow \infty} \lambda_n = s$, where s is any number finite or infinite but not zero. Then by virtue of a well known result we have

$$\lim_{n \rightarrow \infty} \frac{1}{P_n} \sum_1^n \lambda_m p_m = s.$$

Since $s \neq 0$ we get a contradiction by virtue of (4.1). Hence s must be zero so that $\lambda_n \rightarrow 0$ and therefore $\lim_{n \rightarrow \infty} \Delta \lambda_n = 0$. Thus $\Delta \lambda_n \geq 0$. This means that $\{\lambda_n\}$ is a non-increasing sequence and by virtue of the fact that $\lim_{n \rightarrow \infty} \lambda_n = 0$, it follows that $\{\lambda_n\}$ is non-negative and decreasing sequence tending to zero.

We know that if $\Sigma a_n < \infty$ and $\{\beta_n\}$ is any monotonic increasing sequence of positive numbers tending to infinity with n , then $\lim_{n \rightarrow \infty} \frac{1}{\beta_n} \sum_1^n a_m \beta_m = 0$. Taking $\beta_n = 1/\lambda_n$ and applying this result to the series $\Sigma p_n \lambda_n$, which is convergent, we have

$$\lambda_n \sum_1^n \lambda_m p_m 1/\lambda_m \rightarrow 0,$$

that is to say $P_n \lambda_n = o(1)$, as $n \rightarrow \infty$.

This completes the proof of the lemma.

Lemma 2. *If $\{\alpha_n \gamma_n\}$ satisfies the same condition as λ_n in Lemma 1, then*

$$\sum_1^m P_n \gamma_n \Delta \alpha_n = O(1), \quad m \rightarrow \infty$$

where $\{\gamma_n\}$ is a positive non-decreasing sequence such that

$$\Delta \gamma_n = O(p_n \gamma_n / P_n).$$

If we take $p_n = \frac{1}{n+1}$ and $\gamma_n = 1$, we get the following lemma due to PATI ([4]).

Lemma B. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n/n < \infty$, then*

$$\sum_1^m \log(n+1) \Delta \lambda_n = O(1),$$

as $m \rightarrow \infty$.

On the other hand, if we take $\gamma_n = \{\log(n+1)\}^k$, $k \geq 0$ and $p_n = \frac{1}{n+1}$, we obtain the following result of PRASAD and BHATT ([6]).

Lemma C. *If $\{(\log(n+1))^k \alpha_n\}$ satisfies the same condition as λ_n in Lemma A, then*

$$\sum_1^m \{\log(n+1)\}^{k+1} \Delta \alpha_n = O(1), \quad (m \rightarrow \infty).$$

PROOF OF LEMMA 2.

$$\begin{aligned} \sum_1^m \gamma_n \alpha_n p_n &= \sum_1^{m-1} \Delta(\gamma_n \alpha_n) P_n + \gamma_m \alpha_m P_m = \\ &= \sum_1^{m-1} (\gamma_n \Delta \alpha_n + \alpha_{n+1} \Delta \gamma_n) P_n + o(1) = \sum_1^{m-1} \gamma_n P_n \Delta \alpha_n + \sum_1^{m-1} \alpha_{n+1} \Delta \gamma_n P_n + o(1) = \\ &= \sum_1^{m-1} \gamma_n P_n \Delta \alpha_n + O\left(\sum_1^{m-1} p_n \gamma_n \alpha_{n+1}\right) + o(1) = \sum_1^{m-1} \gamma_n P_n \Delta \alpha_n + O(1). \end{aligned}$$

Therefore

$$\sum_1^m \gamma_n P_n \Delta \alpha_n = O(1) \quad (m \rightarrow \infty).$$

Lemma 3. *If $\{\gamma_n \alpha_n\}$ satisfies the same condition as λ_n in Lemma 1, where $\{\gamma_n\}$ is a positive non-decreasing sequence such that $\Delta^2 \frac{1}{\gamma_n} \geq 0$ and*

$$\Delta \gamma_n = O(\gamma_n p_n / P_n),$$

and $\{p_n\}$ is non-increasing sequence, then we have

$$\sum_1^m n P_n \gamma_n \Delta^2 \alpha_n = O(1),$$

and

$$m P_m \gamma_m \Delta \alpha_m = O(1) \quad (m \rightarrow \infty).$$

The following lemmas are the special cases of this result.

Lemma D (PATI [5]). If $\{\lambda_n\}$ satisfies the condition of Lemma A, then

$$m \log(m+1) \Delta \lambda_m = O(1),$$

and

$$\sum_1^m n \log(n+1) \Delta^2 \lambda_n = O(1),$$

as $m \rightarrow \infty$.

Lemma E (BHATT [2]). If $\{(\log(n+1))^k \alpha_n\}$, $k \geq 0$, satisfies the same condition as λ_n in Lemma A, then

$$m \{\log(m+1)\}^{k+1} \Delta \alpha_m = O(1),$$

and

$$\sum_1^m n \{\log(n+1)\}^{k+1} \Delta^2 \alpha_n = O(1),$$

as $m \rightarrow \infty$.

PROOF OF LEMMA 3. We have

$$\begin{aligned} \Sigma' &\equiv \sum_1^m \gamma_n P_n \Delta \alpha_n = \sum_1^{m-1} (n+1) \Delta (\gamma_n P_n \Delta \alpha_n) + (m+1) \gamma_m P_m \Delta \alpha_m = \\ &= \sum_1^{m-1} (n+1) \{\gamma_n P_n \Delta^2 \alpha_n + \Delta \alpha_{n+1} \Delta (\gamma_n P_n)\} + (m+1) \gamma_m P_m \Delta \alpha_m = \\ &= \sum_1^{m-1} (n+1) \gamma_n P_n \Delta^2 \alpha_n + \sum_1^{m-1} (n+1) \Delta \alpha_{n+1} \Delta \gamma_n P_n + \\ &+ \sum_1^{m-1} (n+1) \Delta \alpha_{n+1} \gamma_{n+1} (-p_{n+1}) + (m+1) \gamma_m P_m \Delta \alpha_m = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4. \end{aligned}$$

Applying Lemma 2 we have $\Sigma' = O(1)$ and

$$\Sigma_2 = O\left(\sum_1^{m-1} n p_n \gamma_n P_n (\Delta \alpha_{n+1}) / P_n\right) = O\left(\sum_1^{m-1} P_n \gamma_n \Delta \alpha_{n+1}\right) = O(1),$$

since $\Delta^2 \alpha_n \geq 0$. Similarly

$$\Sigma_3 = O\left(\sum_1^{m-1} P_{n+1} \gamma_{n+1} \Delta \alpha_{n+1}\right) = O(1),$$

so that $\Sigma_1 + \Sigma_4 = O(1)$. Since Σ_1 and Σ_4 are positive the results follow.

5. PROOF OF THEOREM 1. Let $C_n = a_n P_n \lambda_n / \gamma_n$, $\alpha_n = \lambda_n / \gamma_n$,

$$T_n = \sum_0^n C_m \quad \text{and} \quad t_n^* = \frac{1}{P_n} \sum_0^n p_m T_m.$$

Then we have

$$\begin{aligned}
 t_n^* - t_{n+1}^* &= \frac{1}{P_n} \sum_0^n p_m T_m - \frac{1}{P_{n+1}} \sum_0^{n+1} p_m T_m = \\
 &= \Delta(1/P_n) \sum_0^n p_m T_m - p_{n+1} T_{n+1}/P_{n+1} = \\
 &= -\Delta(1/P_n) \sum_0^{n-1} P_m C_{m+1} + \Delta(1/P_n) P_n T_n - \frac{p_{n+1} T_{n+1}}{P_{n+1}} = \\
 &= -\Delta(1/P_n) \sum_0^n P_m C_{m+1}.
 \end{aligned}$$

Now

$$\begin{aligned}
 \sum_0^n P_m C_{m+1} &= \sum_0^n P_m \lambda_{m+1} P_{m+1} a_{m+1}/\gamma_{m+1} = \\
 &= \sum_0^{n-1} \Delta \left(\frac{P_m \lambda_{m+1}}{\gamma_{m+1}} \right) \sum_{\mu=0}^m P_{\mu+1} a_{\mu+1} + \frac{P_n \lambda_{n+1}}{\gamma_{n+1}} \sum_{\mu=0}^n P_{\mu+1} a_{\mu+1} = \\
 &= \sum_0^{n-1} \Delta \left(\frac{P_m \lambda_{m+1}}{\gamma_{m+1}} \right) (\mu_{m+1} P_{m+1} - a_0 P_0) + \frac{P_n \lambda_{n+1}}{\gamma_{n+1}} (\mu_{n+1} P_{n+1} - a_0 P_0) = \\
 &= -a_0 P_0 (P_0 \lambda_1/\gamma_1 - P_n \lambda_{n+1}/\gamma_{n+1}) + \sum_0^{n-1} \Delta(P_m \lambda_{m+1}/\gamma_{m+1}) \mu_{m+1} P_{m+1} - \\
 &\quad - a_0 P_0 P_n \lambda_{n+1}/\gamma_{n+1} + P_n \lambda_{n+1} \mu_{n+1} P_{n+1}/\gamma_{n+1} = \\
 &= \sum_0^{n-1} \Delta(P_m \lambda_{m+1}/\gamma_{m+1}) \mu_{m+1} P_{m+1} + \frac{P_n \lambda_{n+1} \mu_{n+1} P_{n+1}}{\gamma_{n+1}} + O(1) = \\
 &= \sum_0^n \Delta(P_m \lambda_{m+1}/\gamma_{m+1}) \mu_{m+1} P_{m+1} + P_{n+1}^2 \mu_{n+1} \lambda_{n+2}/\gamma_{n+2} + O(1) = \\
 &= \sum_0^{n-1} (m+1) \Delta \{P_{m+1} \Delta(P_m \lambda_{m+1}/\gamma_{m+1})\} \frac{1}{(m+1)} \cdot \sum_0^m \mu_{r+1} + \\
 &\quad + P_{n+1} \Delta(P_n \lambda_{n+1}/\gamma_{n+1}) (n+1) \cdot \frac{1}{(n+1)} \cdot \sum_0^n \mu_{r+1} + \\
 &\quad + P_{n+1}^2 \mu_{n+1} \lambda_{n+2}/\gamma_{n+2} + O(1) = \\
 &= L_1 + L_2 + L_3 + O(1).
 \end{aligned}$$

It is therefore sufficient to prove that

$$\text{Now} \quad \sum \Delta(1/P_n) |L_r| = O(1) \quad (r=1, 2, 3).$$

$$\begin{aligned} \sum_1^m \Delta(1/P_n) |L_1| &= O \left\{ \sum_1^m \Delta(1/P_n) \sum_0^{n-1} (r+1) \gamma_{r+1} |\Delta \{P_{r+1} \Delta(P_r \lambda_{r+1}/\gamma_{r+1})\}| \right\} = \\ &= O \left\{ \sum_0^{m-1} \frac{(r+1) \gamma_{r+1}}{P_{r+1}} |\Delta \{P_{r+1} \Delta(P_r \alpha_{r+1})\}| \right\} = \\ &= O \left\{ \sum_0^{m-1} \frac{(r+1) \gamma_{r+1}}{P_{r+1}} \cdot P_r P_{r+1} \Delta^2 \alpha_{r+1} \right\} + \\ &+ O \left\{ \sum_0^{m-1} \frac{(r+1) \gamma_{r+1}}{P_{r+1}} P_{r+1} p_{r+1} \Delta \alpha_{r+2} \right\} + \\ &+ O \left\{ \sum_0^{m-1} \frac{(r+1) \gamma_{r+1}}{P_{r+1}} \alpha_{r+3} |\Delta p_{r+1} P_{r+1}| \right\} = \\ &= O \left\{ \sum_0^{m-1} (r+1) \gamma_{r+1} P_{r+1} \Delta^2 \alpha_{r+1} \right\} + O \left\{ \sum_0^{m-1} \gamma_{r+1} P_{r+1} \Delta \alpha_{r+1} \right\} + \\ &+ O \left\{ \sum_0^{m-1} (r+1) \lambda_{r+1} |\Delta(p_{r+1} P_{r+1})|/P_{r+1} \right\} = \\ &= O(1) + O \left\{ \sum_0^{m-1} (r+1) \lambda_{r+1} P_{r+1} (\Delta p_{r+1})/P_{r+1} \right\} + \\ &+ O \left\{ \sum_0^{m-1} (r+1) \lambda_{r+1} p_{r+2}^2/P_{r+1} \right\} = \\ &= O(1) + O \left(\sum_0^{m-1} \Delta p_{r+1} \sum_0^r \lambda_s \right) + O \left(\sum_0^{m-1} p_{r+1} \lambda_{r+1} \right) = O(1), \end{aligned}$$

by virtue of lemmas 2 and 3, the hypotheses and the fact that

$$\sum_0^{m-1} \Delta p_r \sum_0^r \lambda_s = \sum_0^m \lambda_r p_r - p_m \sum_0^m \lambda_r =$$

(5.1)

$$= O(1) + O \left(p_m \frac{1}{p_m} \sum_0^m \lambda_r p_r \right) = O(1).$$

Next

$$\begin{aligned}\sum_1^m \Delta(1/P_n) |L_2| &= O \left(\sum_1^m \Delta(1/P_n) P_{n+1} (n+1) \gamma_{n+1} |\Delta P_n \alpha_{n+1}| \right) = \\ &= O \left(\sum_1^m p_{n+1} (n+1) \gamma_{n+1} p_{n+1} \alpha_{n+2}/P_n \right) + \\ &+ O \left(\sum_1^m p_{n+1} (n+1) \gamma_{n+1} P_n (\Delta \alpha_{n+1})/P_n \right) = \\ &= O \left(\sum_1^m p_{n+1} \lambda_{n+1} \right) + O \left(\sum_1^m P_{n+1} \gamma_{n+1} \Delta \alpha_{n+1} \right) = O(1),\end{aligned}$$

by the hypotheses and lemma 2.

Again

$$\begin{aligned}\sum \Delta(1/P_n) |L_3| &\leq \sum_1^m \Delta(1/P_n) \cdot P_{n+1}^2 |\mu_{n+1}| \lambda_{n+2}/\gamma_{n+2} = \\ &= \sum_1^m \frac{P_{n+1}}{\gamma_{n+2}} \lambda_{n+2} P_{n+1} |\mu_{n+1}|/P_n = \sum_1^{m-1} \Delta(p_{n+1} P_{n+1} \alpha_{n+2}/P_n) \sum_0^n |\mu_{r+1}| + \\ &+ (p_{m+1} P_{m+1} \alpha_{m+2}/P_m) \sum_0^m |\mu_{r+1}| = \\ &= O \left(\sum_1^{m-1} (n+1) \gamma_{n+1} |\Delta(p_{n+1} P_{n+1} \alpha_{n+2}/P_n)| \right) + \\ &+ O(p_{m+1} P_{m+1} (m+1) \alpha_{m+2} \gamma_{m+2}/P_m) = L_{31} + L_{32}.\end{aligned}$$

Now

$$L_{32} = O(P_{m+1} \lambda_{m+1}) = o(1) \quad (m \rightarrow \infty).$$

Since

$$\Delta(p_{n+1} P_{n+1}/P_n) = O(p_{n+1}^2/P_n) + O(\Delta p_{n+1}),$$

we have

$$\begin{aligned}L_{31} &= O \left(\sum_1^{m-1} (n+1) \gamma_{n+1} p_{n+1} P_{n+1} (\Delta \alpha_{n+2})/P_n \right) + \\ &+ O \left(\sum_1^{m-1} (n+1) \gamma_{n+1} \alpha_{n+3} |\Delta(p_{n+1} P_{n+1}/P_n)| \right) = \\ &= O \left(\sum_1^{m-1} \gamma_{n+1} P_{n+1} \Delta \alpha_{n+1} \right) + O \left(\sum_1^{m-1} (n+1) \lambda_{n+1} p_{n+1}^2/P_n \right) + \\ &+ O \left(\sum_1^{m-1} (n+1) \lambda_{n+1} \Delta p_{n+1} \right) = \\ &= O(1) + O \left(\sum_1^{m-1} \lambda_{n+1} p_{n+1} \right) + O \left(\sum_1^{m-1} \Delta p_{n+1} \sum_0^n \lambda_r \right) = O(1),\end{aligned}$$

by Lemma 2, the hypotheses and (5.1).

Finally, we have

$$\sum_1^m \Delta(1/P_n) = O(1) \quad (m \rightarrow \infty).$$

This completes the proof of Theorem 1.

6. We deduce the following theorem for $|C, 1|$ summability factors of infinite series.

Theorem 2. *If $\{\lambda_n\}$ is a convex sequence such that $\Sigma \lambda_n < \infty$ and the sequence $\{t_n^1\}$, the $(C, 1)$ mean of $\{na_n\}$, satisfies the condition*

$$|t_n^1| = O(\gamma_n) \quad (C, 1),$$

γ_n being a positive non-decreasing sequence such that $\Delta^2 \frac{1}{\gamma_n} \geq 0$ and $\Delta \gamma_n = O(\gamma_n/n)$, then the series $\Sigma na_n \lambda_n / \gamma_n$ is summable $|C, 1|$.

This generalises a result of PRASAD and BHATT [6] for the summability $|C|$ of order 1. A theorem of TRIPATHI [7] can also be deduced as a corollary from this theorem.

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