

A note on the Jacobson radical of a hemiring

By D. R. LATORRE (Knoxville, Tenn.)

1. *Introduction.* The Jacobson radical of a hemiring (*i.e.*, an additively commutative semiring with zero) was first introduced by BOURNE ([1]) in an informal way, and has subsequently been studied by IIZUKA ([5]) from the point of view of representation theory.

The present paper establishes several new results for the Jacobson radical of a hemiring by considering h -ideals and k -ideals. These ideals are of a more restricted type than the familiar concept of ideal in semirings.

Section 2 contains a brief discussion of subdirect sums of hemirings, and gives analogues of several results well-known in rings.

Our main results are in section 3. We first characterize the Jacobson radical J of a hemiring S in terms of h - and k -ideals, and then prove that if S is a ring, the Jacobson ring and hemiring radicals coincide. An analogue for hemirings of a theorem familiar in ring theory is also proved, from which it follows that any additively regular hemiring S (*i.e.*, for $a \in S$ there exists $x \in S$ such that $a + x + a = a$) for which J is zero is a ring.

2. *Subdirect Sums.* The complete direct sum $\sum_{\alpha} S_{\alpha}$ of a collection of semirings $\{S_{\alpha}\}$, $\alpha \in \Omega$, is defined analogously to the complete direct sum of a collection of rings. For convenience, we shall denote an arbitrary element of $\sum_{\alpha} S_{\alpha}$ by $(s_1, s_2, \dots)$ where $s_{\alpha} \in S_{\alpha}$ and addition and multiplication are defined term by term. However, in adopting this notation we do not intend to imply that the collection $\{S_{\alpha}\}$ is necessarily countable. It is immediate that if each S_{α} is a hemiring, or if each S_{α} is additively regular, then so is $\sum_{\alpha} S_{\alpha}$.

Now let T be a subsemiring of $S = \sum_{\alpha} S_{\alpha}$, $\alpha \in \Omega$, and $t = (t_1, t_2, \dots) \in T$. For each $\alpha \in \Omega$ the mapping $Q_{\alpha}: t \rightarrow t_{\alpha}$ is a homomorphism of T into S_{α} . If, for each α , $TQ_{\alpha} = S_{\alpha}$, we call T a *subdirect sum* of the semirings S_{α} .

A *left semi-ideal* of a semiring S is a non-empty subset I closed under pre-multiplication by elements of S and under addition. A left semi-ideal I of an additively commutative semiring S is called a *left k -ideal* if for each $a \in I$ and $x \in S$, $a + x \in I$ implies that $x \in I$; and I is called a *left h -ideal* of S if $i_1, i_2 \in I$ and $x, z \in S$ with $x + i_1 + z = i_2 + z$ implies $x \in I$. Right semi-, k -, and h -ideals are defined dually and a *two-sided ideal* (or simply *ideal*) of any type is both a left and a right ideal of that type. The concepts of k -ideal and h -ideal are due to HENRIKSEN ([4]) and IIZUKA ([5]), respectively.

By the *Bourne congruence* on an additively commutative semiring S , relative to a given semi-ideal I , we mean the relation $a \equiv b(I)$ if and only if $a + i_1 = b + i_2$ for some $i_1, i_2 \in I$. The *Iizuka congruence* on S , relative to I , is defined by $a[\equiv] b(I)$ if and only if $a + i_1 + z = b + i_2 + z$ for some $i_1, i_2 \in I$ and $z \in S$. Under the usual definitions of addition and multiplication of congruence classes, the Bourne classes and the Iizuka classes form semirings denoted by S/I and $S[\equiv] I$, respectively. The semiring $S[\equiv] I$ is additively cancellative, and both S/I and $S[\equiv] I$ are hemirings in case S is a hemiring. A final definition, again due to Bourne [2], is the following. A semiring S with zero element 0 is said to be *semi-isomorphic* to a semiring T with zero if there is a homomorphism of S onto T with kernel 0. A detailed treatment of k - and h -ideals, semi-isomorphisms, and the semirings S/I and $S[\equiv] I$ can be found in [8].

The proofs of the next two theorems are omitted since they parallel those given in [9] for the corresponding results in rings.

Theorem 2.1. *A semiring S with zero is semi-isomorphic to a subdirect sum T of semirings S_α with zero elements 0_α if and only if there exists homomorphisms φ_α of S onto S_α (for all α) such that, if $0 \neq s \in S$, then $s\varphi_\alpha \neq 0_\alpha$ for some α .*

THEOREM 2.2. *A hemiring S is semi-isomorphic to a subdirect sum of [additively cancellative] semirings $S_\alpha, \alpha \in \Omega$, with zero elements 0_α if and only if for each $\alpha \in \Omega$ there exists a k -ideal [h -ideal] I_α of S such that S/I_α [$S[\equiv] I_\alpha$] is semi-isomorphic to S_α and $\bigcap_{\alpha} I_\alpha = 0$.*

Let T be a semiring with zero that is a subdirect sum of semirings S_α with zero, and let S be a semiring with zero that is semi-isomorphic to T under a mapping σ . As for rings, we call T a *representation* of S as a subdirect sum of the semirings S_α . The mapping φ_α defined by $s\varphi_\alpha = (s\sigma)Q_\alpha$ is a homomorphism of S onto S_α , which may be a semi-isomorphism.

Definition 2.3. A semiring S with zero is called *subdirectly irreducible* if in every representation of S as a subdirect sum of semirings S_α some φ_α is a semi-isomorphism.

By following the proof given for rings in [9] it is readily proved that a hemiring is subdirectly irreducible if and only if the intersection of all its non-zero two-sided k -ideals is non-zero. Unfortunately, there exist examples to show that the term k -ideal cannot here be replaced by h -ideal or semi-ideal. However, if we define a semiring S with zero to be *subdirectly h -irreducible* if in every representation of S as a subdirect sum of additively cancellative semirings S_α , some φ_α is a semi-isomorphism, then a hemiring is subdirectly h -irreducible if and only if the intersection of all its non-zero two-sided h -ideals is non-zero.

Finally, we remark that by again paralleling the proof for rings [9] it can be shown that every hemiring $S \neq 0$ is semi-isomorphic to a subdirect sum of subdirectly irreducible hemirings. Moreover, if the zero of S is an h -ideal then S is semi-isomorphic to a subdirect sum of subdirectly h -irreducible additively cancellative hemirings.

3. The Jacobson Radical. The first to appear [6] among the many equivalent definitions of the Jacobson radical of a ring is the following. The *Jacobson radical* of a ring R is the right ideal generated by the set of all right quasi-regular right

ideals of R . We recall that a right ideal I is called *right quasi-regular* if for $z \in I$ there exists $z' \in R$ such that $z + z' + zz' = 0$.

For convenience, we state Bourne's definition ([1]) of the Jacobson radical of a hemiring. A left [right, two-sided] semi-ideal I of a hemiring S is called *right-semiregular* if for each pair of elements $i_1, i_2 \in I$ there exist elements $j_1, j_2 \in I$ such that

$$i_1 + j_1 + i_1 j_1 + i_2 j_2 = i_2 + j_2 + i_1 j_2 + i_2 j_1.$$

The *Jacobson radical* J of a hemiring S is the right semi-ideal generated by the set of all right-semiregular right semi-ideals of S .

Bourne showed the Jacobson radical, so defined, to be a right-semiregular right semi-ideal of S . Defining left-semiregularity dually, he proved that J is the left semi-ideal generated by the set of all left-semiregular left semi-ideals, and is left-semiregular. In [3] Bourne and Zassenhaus proved that J is a k -ideal, and Iizuka ([5]) proved that J is an h -ideal of S .

Theorem 3.1. *The Jacobson radical J of a hemiring S is the right k -ideal [h-ideal] generated by the set of all right-semiregular right k -ideals [h-ideals] of S .*

PROOF. Let A be the set of all right-semiregular right semi-ideals of S and B the set of all right-semiregular right k -ideals of S . Let $(A)_s$ and $(A)_k$ be respectively the right semi-ideal and right k -ideal generated by A , and let $(B)_k$ be the right k -ideal generated by B . Now $(B)_k \subseteq (A)_k$ since $B \subseteq A$. Since $J = (A)_s$ is a k -ideal it follows that $(A)_k = (A)_s = J$, whence $(B)_k \subseteq J$. But J is a right-semiregular right k -ideal and so $J \subseteq (B)_k$. Thus $J = (B)_k$. Replacing the word k -ideal by h -ideal, and the letter k by h , the above argument gives $J = (B)_h$.

Theorem 3.2. *If R is a ring then the Jacobson hemiring radical J of R coincides with the Jacobson ring radical N of R .*

PROOF. If I is a right-semiregular right k -ideal of R and $i_1 \in I$ then, taking $i_2 = 0$, there exist elements $j_1, j_2 \in I$ such that $i_1 + j_1 + i_1 j_1 = j_2 + i_1 j_2$, that is, $i_1 + (j_1 - j_2) + i_1(j_1 - j_2) = 0$, so that i_1 is right quasi-regular. Since k -ideals in R coincide with ring ideals, it follows that the class A of all right-semiregular right k -ideals is contained in the class C of all right quasi-regular right ideals of R . Thus by Theorem 3.1, $J = (A)_k \subseteq (C)_k$. But $N = (C)_k$, so that $J \subseteq N$. Conversely, if we can show that every right quasi-regular right ideal of R is right-semiregular, then $C \subseteq A$ and so $N \subseteq J$. Thus let I be a right quasi-regular right ideal of R and let $i_1, i_2 \in I$. Then $z = i_1 - i_2 \in I$, so there is an element $z' \in R$ such that $z + z' + zz' = 0$, i.e., $(i_1 - i_2) + z' + (i_1 - i_2)z' = 0$. Therefore $i_1 + z' + i_1 z' = i_2 + i_2 z'$, whence $i_1 + z' + i_1 z' + i_2 0 = i_2 + 0 + i_1 0 + i_2 z'$. Now $z' = -(z + zz') \in I$. Hence there are elements $j_1 = z'$ and $j_2 = 0$ such that $i_1 + j_1 + i_1 j_1 + i_2 j_2 = i_2 + j_2 + i_1 j_2 + i_2 j_1$, so that I is right-semiregular.

Since, for any homomorphism φ of a hemiring S onto a hemiring T , the image of a right-semiregular right semi-ideal under φ is again such an ideal, it follows that φ maps the Jacobson radical of S into that of T . Hence, it is fairly immediate that a subdirect sum of semirings S_α , each of which has zero Jacobson radical, has zero Jacobson radical. These remarks will be used in our next theorem.

The term *primitive*, well-known in rings, has been defined for hemirings by Iizuka ([5]). Theorem 6 of his paper states that *the Jacobson radical of any hemiring*

S that is not a radical hemiring (i.e., $J \neq S$) is the intersection of all primitive h -ideals of S . Using another of Iizuka's results, one can prove that any ring primitive in the hemiring sense is also primitive in the ring sense. We omit the proof since to include it here would necessitate a brief exposition of parts of Iizuka's paper. These remarks bring us to the next theorem.

Theorem 3.3. *If a hemiring $S \neq 0$ has zero Jacobson radical then S is semi-isomorphic to a subdirect sum of primitive hemirings. Conversely, if a hemiring S is semi-isomorphic to a subdirect sum of additively cancellative primitive hemirings then S has zero Jacobson radical.*

PROOF. If $S \neq 0$ has zero Jacobson radical then the intersection of all primitive h -ideals I_α of S is zero. Since each such h -ideal is also a k -ideal, it follows from Theorem 2.2 that S is semi-isomorphic to a subdirect sum of the hemirings S/I_α , each of which is primitive since each I_α is primitive. Conversely, suppose the hemiring S is semi-isomorphic to a subdirect sum T of additively cancellative primitive hemirings S_α . Since each S_α is additively cancellative, the zero 0_α of each is an h -ideal. Therefore, since each S_α is primitive and $S_\alpha \cong S_\alpha/0_\alpha$, 0_α is a primitive h -ideal. It follows that each S_α has zero Jacobson radical, whence, by our earlier remarks, T has zero Jacobson radical and so does S .

Theorem 3.3 is an analogue for hemirings of the familiar theorem for rings due to Jacobson ([7]) stating that a ring $R \neq 0$ has zero Jacobson radical if and only if R is isomorphic to a subdirect sum of primitive (in the ring sense) rings. We have been unable to delete the hypothesis of additive cancellation since, in general, the zero of a hemiring need not be an h -ideal. However, for additively regular hemirings we have the following result, the immediate corollary of which will be useful to us elsewhere.

Theorem 3.4. *An additively regular hemiring $S \neq 0$ has zero Jacobson radical if and only if S is semi-isomorphic to a subdirect sum (in the ring sense) of rings primitive in the ring sense.*

PROOF. Suppose the hemiring $S \neq 0$ is semi-isomorphic to a subdirect sum T (in the ring sense) of rings primitive in the ring sense. Since T is a ring, Theorem 2.4 of [8] shows that S is a ring isomorphic to T . By the result mentioned above, S has Jacobson radical 0.

Conversely, suppose the additively regular hemiring $S \neq 0$ has zero Jacobson radical. Then by the result of Iizuka, the intersection of all h -ideals I_α of S is zero, whence it follows from Theorem 2.2 that S is semi-isomorphic to a subdirect sum T of the hemirings S/I_α . By Corollary 2.8 of [8], each S/I_α is a ring, and, since I_α is a primitive h -ideal of S , S/I_α is a ring primitive in the hemiring sense. Thus, using a result stated earlier, each S/I_α is a ring primitive in the ring sense. The proof will be complete when we show that T is actually a subring of the complete direct sum of the rings S/I_α . To this end, we denote by v_α , for each α , the natural homomorphism of S onto S/I_α . Since $\bigcap_\alpha I_\alpha = 0$, the proofs of Theorems 2.1 and 2.2 show that $T = \{(sv_1, sv_2, \dots) : s \in S\}$. Since each I_α is a k -ideal, I_α is the zero element 0_α of S/I_α , whence the zero of T is $(I_1, I_2, I_3, \dots)$. Now if $s \in S$ and s' is an additive inverse of s in S , then $s'v_\alpha$ is an additive inverse of sv_α in S/I_α . Thus,

for any element $(sv_1, sv_2, sv_3, \dots)$ in T , an additive inverse in T is $(s'v_1, s'v_2, s'v_3, \dots)$. Since $s + s'$ is an additive idempotent in S , its image is an additive idempotent in S/I_α , whence $(s + s')v_\alpha = 0_\alpha = I_\alpha$ since S/I_α is a ring. Therefore, if $(sv_1, sv_2, sv_3, \dots) \in T$ we have $(sv_1, sv_2, sv_3, \dots) + (s'v_1, s'v_2, s'v_3, \dots) = (I_1, I_2, I_3, \dots)$, so that $(sv_1, sv_2, sv_3, \dots)$ has a group inverse in T with respect to addition. Thus T is a ring.

Corollary 3.5. *An additively regular hemiring with zero Jacobson radical is a ring.*

PROOF. The proof of Theorem 3.4 shows S is semi-isomorphic to a ring T , whence it follows that S itself is a ring.

Bibliography

- [1] S. BOURNE, The Jacobson radical of a semiring, *Proc. Nat. Acad. Sci., U.S.A.* **37** (1951), 163—170.
- [2] S. BOURNE, On the Homomorphism Theorem for Semirings, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 118—119.
- [3] S. BOURNE and H. ZASSENHAUS, On the Semiradical of a Semiring, *Proc. Nat. Acad. Sci., U.S.A.* **44** (1958), 907—914.
- [4] M. HENRIKSEN, Ideals in semirings with commutative addition, *Amer. Math. Soc. Notices* **6** (1958), 321.
- [5] K. IIZUKA, On the Jacobson radical of a semiring, *Tohoku Math. J. (2)* **11** (1959), 409—421.
- [6] N. JACOBSON, The radical and semi-simplicity for arbitrary rings, *Amer. J. Math.* **67** (1945), 300—320.
- [7] N. JACOBSON, Structure of rings, *Providence, R. I.*, 1956.
- [8] D. R. LATORRE, On h -ideals and k -ideals in hemirings, *Publ. Math. Debrecen* **12** (1965), 219—226.
- [9] N. H. MCCOY, *The theory of rings*, New York, 1964.

(Received September 29, 1965.)