Generalized complements in modular lattices

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1. Introduction

In the paper [3], published in 1953, I introduced a generalization for the lattice theoretical complement and established some fundamental connections of this new concept with the original one and with the relative complementation, respectively. The aim of the present paper is to add, firstly, some general remarks to the earlier results and, afterwards, to investigate the connections between the generalized complements and the relative complements of the same element in modular lattices.

2. Preliminary remarks and lemmas

Let a, u, v be arbitrary elements of a lattice L. By a *(generalized) complement of a with respect to the elements u, v* or, briefly, by a *(u, v)-complement of a* we mean any element a' of L such that

$$a \cap a' \leqq u, \quad a \cup a' \geqq v. \tag{1}$$

If, in particular, the equality sign holds instead of $\leqq$ and $\geqq$ in (1), then a' is called a *relative complement of a with respect to the elements u, v* or, briefly, a *relative complement of a in $[u, v]$*, where $[u, v]$ means the set of all $x \in L$ with $u \leqq x \leqq v$. (As it is well-known, $a' \in [u, v]$ holds in this case indeed.) For other definitions and notations see [1] or [2].

Remark 1. *Let a, u, v be any elements of a lattice L. Then the set of all (u, v)-complements of a is a convex subset of L. Moreover, this set composes a sublattice, too, if L is distributive.*

For, if a' and a'' are (u, v)-complements of a such that $a' \leqq a''$ and x is any element of $[a', a'']$, then we have

$$a \cap x \leqq a \cap a'' \leqq u, \quad a \cup x \geqq a \cup a' \geqq v.$$

Hence, also the element x is a (u, v)-complement of a. Moreover, if L is distributive, then

$$a \cap (a' \cap a'') \leqq a \cap a' \leqq u,$$

$$a \cup (a' \cap a'') = (a \cup a') \cap (a \cup a'') \geqq v \cap v = v$$

for any (u, v)-complements of a. This means that $a' \cap a''$ and, by the duality principle, also $a' \cup a''$ are (u, v)-complements of a.

Remark 2. *Let a, u, v be elements of a lattice L such that $u \leqq a \leqq v$. A complement a' of the element a with respect to u and v is a relative complement of a with respect to the same elements if and only if a' is contained in $[u, v]$.*

In fact, if $a' \in [u, v]$ then we have

$$a \cap a' \geqq u, \quad a \cup a' \leqq v.$$

These imply, by the inequalities in (1), that $a \cap a' = u$ and $a \cup a' = v$ indeed. The converse (the "only if") statement is trivial.

The following lemma concerns (generalized) complements which are comparable with some relative complement.

Lemma 1. *Let a, u, v be elements of a lattice L and a' a complement of a with respect to u and v. If $u \leqq a \leqq v$ and there exists a relative complement r of a in $[u, v]$ such that $r \leqq a'$, then*

$$a \cap a' = u. \tag{2}$$

Similarly, $r \geqq a'$ implies

$$a \cup a' = v. \tag{3}$$

For, if r is a relative complement of a such that $r \leqq a'$, then

$$u = a \cap r \leqq a \cap a' \leqq u,$$

that is, $a \cap a' = u$. Similarly, in case $r \geqq a'$ we get $a \cup a' = v$.

The lemma which we have just now proved serves as a basis of the following definition: a (u, v)-complement a' of the element a will be called a *semi-relative complement of a with respect to u and v* if there exist a relative complement r of a, just like with respect to u and v, such that a' and r are comparable. Clearly, an element a has semi-relative complements with respect to the elements u, v only if it is contained in the interval $[u, v]$.

The set of all semi-relative complements of a with respect to u and v will be denoted by $C_a(u, v)$. If $a \notin [u, v]$ then, of course, $C_a(u, v)$ is empty.

In the next section we will make use also the following

Lemma 2. *Let L be a lattice and let a, u, v, p, q be elements of L such that $p < u$ and $q > v$. If the elements x and y are relative complements of a with respect to the elements u, v and p, q, respectively, then x and y are incomparable.*

Suppose $x \leqq y$. Then $u = a \cap x \leqq a \cap y = p$, in contradiction to our assumption $u > p$. Similarly, $x \geqq y$ would imply $v \geqq q$, again a contradiction.

3. The set of (u, v)-complements

Following the terminology introduced in [3], a lattice L will be called *complemented* (in generalized sense) if, given arbitrary elements a, u, v of L, there exists at least one (u, v)-complement of a.

Theorem 1. *Let L be a complemented lattice and let a, u, v be elements of L. If neither u is a least element in L, nor v a greatest one, then there exists a complement of a with respect to the elements u, v which is no semi-relative complement of a with respect to the same elements.*

PROOF. If u is no least element and v is no greatest element in L, then there exist elements u_1, v_1 in L such that $u_1 < u$ and $v_1 > v$. Let y be a (u_1, v_1)-complement of a and denote p and q the elements $a \cap y$ and $a \cup y$, respectively. Then

$$p = a \cap y \leqq u_1 < u, \quad q = a \cup y \geqq v_1 > v.$$

Hence, for the elements a, u, v, p, q, y the assumptions of Lemma 2 are satisfied. Consequently, each relative complement x of a in $[u, v]$ is incomparable with y. Since y is a complement of a with respect to the elements u, v too, the theorem is proved.

From now on we shall be concerned with modular lattices. Firstly we show that also the conversion of Lemma 1 holds in this case:

Theorem 2. *Let a, u, v $(u \leqq v)$ be elements of a modular lattice L and let a' be a (u, v)-complement of a. Then a' is a semi-relative complement of a with respect to u and v if and only if at least one of the equations* (2) *and* (3) *holds.*

PROOF. The "only if" part of the theorem is true by Lemma 1. In order to prove the "if" part, suppose (2) to be satisfied. Then a' is a complement of a in the sublattice $S = [u, i]$ with $i = a \cup a'$. The lattice S is itself modular ([1], p. 65, or [2], p. 87) and $S \ni v$. Thus, by a theorem due to VON NEUMANN ([1], p. 114, or [2], p. 112), the element $s = (u \cup a') \cap v = a' \cap v$ is a relative complement of a in $[u, v]$ and $s \leqq a'$. Similarly, (3) implies the existence of a relative complement t of a in $[u, v]$ with $t \geqq a'$.

The five-element non-modular lattice shows, that, without assuming the modularity, the statement of Theorem 2 does not remain valid.

The main purpose of this section is to clear the structure of the sets $C_a(u, v)$ for the case when the lattice is modular.

Theorem 3. *Let L be a modular lattice and let a, u, v be elements of L such that $u \leqq a \leqq v$. Then there exists, to each (u, v)-complement a' of a, at most one relative complement of a in $[u, v]$ which is comparable with a'.*

PROOF. Suppose that there exist two relative complements of a in $[u, v]$, say r_1 and r_2, which are both comparable with a'. Since, by the modularity, r_1 and r_2 are incomparable ([1], p. 66, or [2], p, 90), either

$$r_1, r_2 \leqq a' \tag{4}$$

or the dual of (4) holds. According to the duality principle, it suffices to consider the case when (4) is satisfied.

In this case we have

$$a \cap (r_1 \cup r_2) \leqq a \cap a' \leqq u. \tag{5}$$

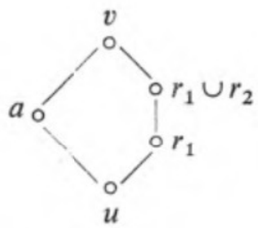
But, on the other hand,

$$a \cap (r_1 \cup r_2) \geqq a \cap r_1 = u. \tag{6}$$

From (5) and (6) we infer immediately

$$a \cap (r_1 \cup r_2) = u.$$

This means that the elements u, a, r_1, $r_1 \cup r_2$ and v compose in L a non-modular sublattice with the following diagram:



By this contradiction our theorem is proved.

Let L be a modular lattice and a, u, v any elements of L such that $u \leqq a \leqq v$. If r is a relative complement of a in $[u, v]$, then we denote by C_r the set of those semi-relative complements of a with respect to u and v, which are comparable with r. The family of all C_r, where r runs over all relative complements of a in $[u, v]$, determine by Theorem 3 a partition of the set $C_a(u, v)$. Moreover, any pair of elements of $C_a(u, v)$, belonging to different classes of this partition, are incomparable by the next theorem:

Theorem 4. *Let L be a modular lattice and a, u, v elements of L such that $u \leqq a \leqq v$. Let r', r'' $(r' \neq r'')$ be relative complements and a', a'' complements of a with respect to the elements u, v. If a' is comparable with r' and a'' with r'', then a' and a'' are incomparable.*

PROOF. Suppose that a', r' and a'', r'' are comparable pairs. Then one of the following three cases occurs:

(i) $a' \leqq r'$ and $a'' \leqq r''$;
(ii) $a' \geqq r'$ and $a'' \leqq r''$;
(iii) $a' \geqq r'$ and $a'' \geqq r''$.

In cases (i) and (iii), the (u, v)-complements a' and a'' are incomparable by Theorem 3. Furthermore, in Case (ii) the inequality $a' \leqq a''$ would imply $r' \leqq r''$ what is impossible, because the lattice L is modular. Thus, in order to finish the proof, we have only to show that also $a' \geqq a''$ is impossible in Case (ii).

Suppose (ii) and $a' \geqq a''$. Then we obtain

$$r' \leqq r' \cup a'' \leqq a'$$

and

$$r' \leqq r' \cup a'' \leqq r' \cup r'' \leqq v \cup v = v,$$

whence

$$r' \leqq r' \cup a'' \leqq a' \cap v. \tag{7}$$

Thus the equation $r' = a' \cap v$ would imply $r' = r' \cup a''$, i.e. $r' \geqq a''$ which is impossible by Theorem 3 (because $r'' \geqq a''$). Consequently

$$r' < a' \cap v. \tag{8}$$

Aqain by $r'' \geqq a''$ and by Lemma 1, we have $a \cup a'' = v$. Hence, by (7),

$$v = v \cup a'' = (a \cup r') \cup a'' = a \cup (r' \cup a'') \leqq a \cup (a' \cap v) \leqq v \cap v = v,$$

that is,

$$a \cup (a' \cap v) = v. \tag{9}$$

On the other hand, by $r' \leqq a'$ and by Lemma 1, we have $a \cap a' = u$. Applying this equation, we get

$$a \cap (a' \cap v) = (a \cap a') \cap v = u \cap v = u,$$

that is,

$$a \cap (a' \cap v) = u. \tag{10}$$

By (9) and (10) the element $a' \cap v$ is a relative complement of a in $[u, v]$. But this conclusion is in contradiction to (8), because in modular lattice no element has two distinct comparable relative complements with respect to the same elements.

Thus Theorem 4 is proved.

We conclude this paper by an almost obvious

Remark. *Let a, u, v be elements of a lattice such that $u \leqq a \leqq v$ and let r be a relative complement of a in $[u, v]$. If L is modular, then there exists no (u, v)-complement of a in $[u, r]$ or $[r, v]$ different from r itself.*

For, if a' is a (u, v)-complement of a such that $r < a' \leqq v$, then $a \cap a' = u$ by Lemma 1 and $a \cup a' = v$ trivially. Thus a' would be also a relative complement of a in $[u, v]$ and this is in contradiction to the fact L is modular and $r \neq a'$. Similarly, if we suppose $u \leqq a' \cap r$, then we get again to this contradiction.

References

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