

Note on the spectra of finite permutation matrices

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§ 1. *Introduction.* The spectrum of the special $n \times n$ permutation matrix

$$(1) \quad C_n = [\delta_{\nu, \mu-1}] \quad \delta_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha \equiv \beta \pmod{n} \\ 0 & \text{for } \alpha \not\equiv \beta \pmod{n} \end{cases}$$

i.e. the *circulant matrix* with the first row $(0, 1, 0, \dots, 0)$ is already known from the classical determinant theory (see for example [1]). The matrix C_n has only simple eigenvalues, hence its spectral decomposition is uniquely determined. Let ε_n^k denote the n -th root of unity $\varepsilon_n^k = \exp i \frac{2k\pi}{n}$ then the λ_k eigenvalue and the eigenvector u_k belonging to λ_k are

$$(2) \quad \left. \begin{aligned} \lambda_k &= \varepsilon_n^k \\ u_k^T &= \frac{1}{\sqrt{n}} (1, \varepsilon_n^k, \varepsilon_n^{2k}, \dots, \varepsilon_n^{(n-1)k}) \end{aligned} \right\} \quad k = 0, 1, \dots, n-1.$$

The raised T is used to denote transposition.

In the general case that is if a permutation matrix P_n has also multiple eigenvalues, one has certain freedom to choose the linearly independent eigenvectors belonging to the same multiple eigenvalue. The present paper is intended to give a formula for a spectral decomposition of any permutation matrix P_n , as simple as (2) that is a formula such that *the nonzero components of an eigenvector belonging to an eigenvalue contain the powers of this eigenvalue only.*

In discussing the spectra of the permutation matrices it seems very helpful to use some group theoretical relations.

§ 2. The $n \times n$ permutation matrices P_n give an n -dimensional real unitary representation of the symmetric group S_n . This representation is called the *permutation representation* (see [2] p. 79).

Let p denote a permutation $p \in S_n$ and $\alpha^{[p]}$ the image of the element α ($\alpha = 1, 2, \dots, n$) under the permutation p , hence

$$(3) \quad p = \begin{pmatrix} 1 & 2 & \dots & n \\ 1^{[p]} & 2^{[p]} & \dots & n^{[p]} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha^{[p]} \end{pmatrix}$$

We read the product pq from left to right, that is $\alpha^{[pq]} = (\alpha^{[p]})^{[q]}$. Thus, in the above representation the permutation (3) is represented by the matrix

$$(4) \quad P_n(p) = [\delta_{v^{[p]}\mu}] \quad v, \mu = 1, 2, \dots, n.$$

Here

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta \end{cases}$$

is the Kronecker symbol. With this notation the matrix C_n in (1) can be written as

$$(1a) \quad C_n = P_n(c) = [\delta_{v^{[c]}\mu}]$$

where the permutation c consists of the single cycle

$$c = (1, 2, \dots, n).$$

Postmultiplication by $P_n(p)$ of any $n \times n$ matrix M results in the rearrangement of the columns of the matrix M corresponding to the permutation p , while the premultiplication by $P_n(p)$ rearranges the rows of M corresponding to the permutation p^{-1} .

Two permutations $p, q \in S_n$ are conjugate if and only if both contain disjunct cycles of the same number and the same length resp. Suppose p and q are conjugate, i.e.

$$(5) \quad q = s^{-1}ps \quad \text{with } s \in S_n$$

then the cyclic form of q is obtained by replacing by $\alpha^{[s]}$ each element α in the cyclic form of p .

From (5) we have the orthogonal similarity relation between the matrices $P_n(p)$ and $P_n(q)$:

$$(6) \quad P_n(q) = P_n(s^{-1})P_n(p)P_n(s) = P_n^T(s)P_n(p)P_n(s).$$

Hence, obviously

Lemma. Suppose $q = s^{-1}ps$ and let u be an eigenvector of $P_n(p)$ corresponding to the eigenvalue λ , i.e.

$$(7) \quad P_n(p)u = \lambda u$$

then the vector

$$(8) \quad v = P_n^T(s)u$$

is an eigenvector of $P_n(q)$ corresponding to the eigenvalue λ . The relations between the components of the two vectors $u^T = (u_1, u_2, \dots, u_n)$ and $v^T = (v_1, v_2, \dots, v_n)$ are given by

$$(9) \quad v_{v^{[s]}} = u_v \quad v = 1, 2, \dots, n.$$

§ 3. It follows from the above discussion that for determining a spectral decomposition of any permutation matrix $P_n(p)$ ($p \in S_n$) it is enough to choose any single permutation p in each conjugate class K of S_n and determine a spectral decomposition of the matrix $P_n(p)$. Moreover in each conjugate class K we can find a permutation p_0 such that the matrix $P_n(p_0)$ decomposes into the direct sum of

circulant matrices. For illustration let us consider the permutation p_0 which consists of m disjunct cycles of length $l_1, l_2, \dots, l_m$

$$(10) \quad p_0 = (12 \dots l_1) \dots (a_k(a_k+1) \dots (a_k+l_k-1)) \dots (a_m(a_m+1) \dots (a_m+l_m-1))$$

$$a_k = 1 + \sum_{v=1}^{k-1} l_v \quad (k = 2, \dots, m); \quad a_1 = 1; \quad a_m + l_m - 1 = n.$$

Obviously the permutation matrix $P_n(p_0)$ decomposes into the direct sum of m circulants of order $l_1, l_2, \dots, l_m$ as

$$(11) \quad P_n(p_0) = C_{l_1} + C_{l_2} + \dots + C_{l_m} = \text{diag}(C_{l_1}, C_{l_2}, \dots, C_{l_m}).$$

Thus a spectral decomposition of $P_n(p_0)$ is known by (2): The cycle $(a_k(a_k+1) \dots (a_k+l_k-1))$ contributes to the spectra of $P_n(p_0)$ the eigenvector u_{kv} corresponding to the eigenvalue $\varepsilon_{l_k}^v = \exp i \frac{2v\pi}{l_k}$ where

$$(12) \quad u_{kv}^T = \frac{1}{\sqrt{l_k}} (\underbrace{0, 0, \dots, 0}_{a_k-1}, \underbrace{1}_{a_k}, \underbrace{\varepsilon_{l_k}^v}_{a_k+1}, \dots, \underbrace{\varepsilon_{l_k}^{(l_k-1)v}}_{a_k+l_k-1}, 0, \dots, 0) \quad v = 0, 1, \dots, l_k-1;$$

$$k = 1, 2, \dots, m.$$

Now suppose we want to determine a spectrum of the permutation matrix $P_n(q)$ where the permutation q is conjugate permutation to (10)

$$q = s^{-1} p_0 s \quad s \in S_n.$$

Then the cycle decomposition of q is

$$(13) \quad q = (1^{[s]} 2^{[s]} \dots l_1^{[s]}) \dots (a_k^{[s]}(a_k+1)^{[s]} \dots (a_k+l_k-1)^{[s]}) \dots (a_m^{[s]}(a_m+1)^{[s]} \dots n^{[s]})$$

$$= (\alpha_1 \alpha_2 \dots \alpha_{l_1}) \dots (\kappa_1 \kappa_2 \dots \kappa_{l_k}) \dots (\mu_1 \mu_2 \dots \mu_{l_m}).$$

By the lemma and the spectrum (12) of the matrix $P_n(p_0)$ in (11) a spectral decomposition of $P_n(q)$ can be written down without calculation to and we have the

Theorem. Suppose $q \in S_n$ is the permutation of the cycle decomposition (13), then a spectral decomposition of the permutation matrix $P_n(q)$ can be chosen as follows. Each cycle

$$(\kappa_1 \kappa_2 \dots \kappa_{l_k})$$

contributes to the spectrum the eigenvector v_{kv} corresponding to the eigenvalue $\varepsilon_{l_k}^v = \exp i \frac{2v\pi}{l_k}$ ($v = 0, 1, \dots, l_k-1$) where the vector v_{kv} has only l_k nonzero components namely those, corresponding to the indices κ_j , denoted by $v_{kv}(\kappa_j)$ ($j = 1, 2, \dots, l_k$), when the values of these components are

$$v_{kv}(\kappa_j) = \frac{1}{\sqrt{l_k}} \varepsilon_{l_k}^{(j-1)v} \begin{cases} j = 1, 2, \dots, l_k \\ v = 0, 1, \dots, l_k-1 \\ k = 1, 2, \dots, m. \end{cases}$$

It is worthy of note that any symmetric permutation matrix represents a permutation which contains at most transpositions i.e. the lengths of its disjunct cycles are one or two. Thus a symmetric permutation matrix has only two different eigenvalues ± 1 and the corresponding eigenvectors can be chosen such that they contain at most two nonzero components.

§ 4. Let us see now an example. We want to determine a spectral decomposition of the permutation matrix

$$P_7(q) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The cyclic form of q is

$$q = (143)(26)(57).$$

The cycle (143) contribute to the spectrum the eigenvalues $\varepsilon_3^v = \exp i \frac{2v\pi}{3}$ ($v=0, 1, 2$) with the eigenvectors

$$\lambda = 1 = \varepsilon_3^0; \quad \frac{1}{\sqrt{3}}(1, 0, 1, 1, 0, 0, 0)$$

$$\lambda = \varepsilon_3; \quad \frac{1}{\sqrt{3}}(1, 0, \varepsilon_3^2, \varepsilon_3, 0, 0, 0) = \frac{1}{\sqrt{3}}(1, 0, \bar{\varepsilon}_3, \varepsilon_3, 0, 0, 0)$$

$$\lambda = \varepsilon_3^2 = \bar{\varepsilon}_3; \quad \frac{1}{\sqrt{3}}(1, 0, \varepsilon_3^4, \varepsilon_3^2, 0, 0, 0) = \frac{1}{\sqrt{3}}(1, 0, \varepsilon_3, \bar{\varepsilon}_3, 0, 0, 0).$$

The contribution of the cycles (26) and (57) to the spectrum are the eigenvalues ± 1 with the eigenvectors

$$\lambda = 1; \quad \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, 1, 0) \quad \text{and} \quad \frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, 0, 1)$$

$$\lambda = -1; \quad \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, -1, 0) \quad \text{and} \quad \frac{1}{\sqrt{2}}(0, 0, 0, 0, 1, 0, -1)$$

respectively.

References

- [1] E. PASCAL, I determinanti. *Milano*, 1897.
- [2] H. WIELANDT, Finite permutation groups. *New York*, 1964.

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