Kolmogorov automorphisms in σ-finite measure spaces

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Introduction

It is well known (see [3]) that for finite measure spaces Kolmogorov automorphisms are ergodic. The aim of this paper is to extend the concept of Kolmogorov automorphisms and the above theorem to σ-finite measure spaces.

Notation

Let X be an arbitrary space, ε be a σ-algebra of subsets of X, μ a σ-finite measure on (X, ε) and T a measure preserving automorphism on (X, ε, μ). By a σ-algebra α we mean a sub-σ-algebra of ε and by a set A we mean a subset of X such that $A \in \varepsilon$. For any set A we put

$$\varepsilon_A = \{B: B \in \varepsilon,\ B \subseteqq A\}$$

and define a mesure μ_A on (A, ε_A) by putting

$$\mu_A(B) = \mu(B) \quad \text{for} \quad B \in \varepsilon_A.$$

We define the induced automorphism S_A by putting

$$S_A(x) = \{T^j x\colon T^i x \in A,\ T^j x \notin A,\ 1 \leqq j \leqq i-1 \quad \text{for} \quad x \in A\}$$

and lastly for any σ-algebra α and any set A we put

$$\alpha_A = \{B\colon \text{there exists a } C \in \alpha \text{ such that } B = A \cap C\}.$$

Preliminaries

We say that a set A is a wandering set if $A \cap T^{-i}A = \emptyset$ for $i = 1, 2, \ldots$. It is well known (see [1]) that if (X, ε, μ, T) has no wandering sets of positive measure then for all $A \in \varepsilon$ we have

$$\bigcup_{i=1}^{\infty} T^{-i}A = \bigcup_{i=0}^{\infty} T^{-i}A$$

up to a set of measure zero and that

$$A = \bigcup_{i=1}^{\infty} A_i$$

up to a set of measure zero where

$$A_i = \{x \colon x \in A,\ T^i x \in A,\ T^j x \notin A,\ 1 \leqq j \leqq i-1\}.$$

Lemma 1. *If there are no wandering sets of positive measure in (X, ε, μ, T) and α is any σ-algebra such that $\alpha \leqq T\alpha$ then for all sets A with $0 < \mu(A)$ we have $\alpha_A \leqq S_A \alpha_A$.*

PROOF. For any $B \in \alpha_A$ (and hence to α) we put $B_k = T^k A \cap B - \bigcup_{j=1}^{k-1} B_j$, $k = 1, 2, \ldots$ Then $B_k \subseteqq B$ and $S_A^{-1} B_k = T^{-k} B_k$ for all k. If $C = B - \bigcup_{k=1}^{\infty} B_k$ then $C \cap T^i C = \emptyset$ for $i = 1, 2, \ldots$ and hence we must have $\mu(C) = 0$ i.e. $B = \bigcup_{k=1}^{\infty} B_k$ up to a set of measure zero. Further $B_k \in T^k \alpha$ for $k = 1, 2, \ldots$ and so up to a set of measure zero we have

$$B = \bigcup_{k=1}^{\infty} B_k = S_A S_A^{-1} \bigcup_{k=1}^{\infty} B_k = S_A \bigcup_{k=1}^{\infty} T^{-k} B_k.$$

Now $T^{-k} B_k \in \alpha$, $T^{-k} B_k \subseteqq A$ and so we get that $\bigcup_{k=1}^{\infty} T^{-k} B_k \in \alpha_A$ i.e. $B \in S_A \alpha_A$. But B was any set in α_A and so we conclude that $\alpha_A \leqq S_A \alpha_A$.

Corollary 1. With the hypothesis of the lemma $\alpha_A \leqq (T\alpha)_A \leqq S_A \alpha_A$.

PROOF. Since $\alpha \leqq T\alpha$ we have $\alpha_A \leqq (T\alpha)_A$. The second inequality is proved by taking $B \in (T\alpha)_A$ in the proof of the lemma.

Corollary 2. With the hypothesis of the lemma if $\bigvee_{i=-\infty}^{\infty} T^i \alpha = \varepsilon$ then $\bigvee_{i=-\infty}^{\infty} S_A^i \alpha_A = \varepsilon_A$.

PROOF. $\varepsilon_A = \left(\bigvee_{i=-\infty}^{\infty} T^i \alpha \right)_A \leqq \left(\bigvee_{i=-\infty}^{\infty} S_A^i \alpha_A \right) \leqq \varepsilon_A.$

Lemma 2. *If there are no wandering sets of positive measure in (X, ε, μ, T) α is a σ-algebra such that $\alpha \leqq T\alpha$, $\bigvee_{i=-\infty}^{\infty} T^i \alpha = \varepsilon$, $A \in \alpha$ satisfies $0 < \mu(A) < \infty$ and $B \in \varepsilon_A$ then $S_A B = B$ implies $B \in \alpha_A$.*

PROOF. For each $k = 1, 2, \ldots$ there exists B_k, n_k such that $B_k \in S_A^{n_k} \alpha_A$, $\mu(B \triangle B_k) < 2^{-k}$. But $B = S_A B$ and so

$$\mu(B \triangle S_A^{-n_k} B_k) = \mu\{S^{-n_k}(B \triangle B_k)\} = \mu(B \triangle B_k) < 2^{-k}$$

If $C_k = S^{-n_k} B_k$ for $k = 1, 2, \ldots$ then $C_k \in \alpha_A$ for each k and

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} C_k$$

up to a set of measure zero. Hence $B \in \alpha_A$ as required.

Lemma 3. *If there are no wandering sets of positive measure in* (X, ε, μ, T), $A \in \varepsilon$, $\mu(A) > 0$ *and* S_A *is ergodic, then* T *is ergodic in* $(B, \varepsilon_B, \mu_B)$ *where* $B = \bigcup_{i=1}^{\infty} T^{-i} A$.

PROOF. See S. KAKUTANI ([2]).

The main resolt

We say that T is a Kolmogorov automorphism if there exists a σ-algebra α such that

(i) $\alpha \leqq T\alpha$

(ii) $\bigvee_{i=-1}^{\infty} T^i \alpha = \varepsilon$

(iii) $\bigwedge_{i=-1}^{\infty} T^i \alpha = \nu$, where ν is the σ-algebra consisting of the two sets: $\emptyset$ and X.

(iv) for at least one $A \in \alpha$ we have $0 < \mu(A) < \infty$.

Theorem. *If there are no wandering sets of positive measure in* (X, ε, μ, T) *and if* T *is a Kolmogorov automorphism then* T *is ergodic.*

PROOF. If α is a σ-algebra satisfying (i)—(iv) then if $A \in \alpha$, $0 < \mu(A) < \infty$ we have $\bigcup_{i=1}^{\infty} T^i A$ to be invariant and hence by (iii) we get $\bigcup_{i=1}^{\infty} T^i A = X$. Thus we can find A_n, $n = 1, 2, \dots$ such that $A_n \in \alpha$, $0 < \mu(A_n) < \infty$ each n, $A_n \cap A_m = \emptyset$ if $n \neq m$ and $\bigcup_{n=1}^{\infty} A_n = X$. We write $\varepsilon_n, \mu_n, S_n, \alpha_n$ for $\varepsilon_{A_n}, \mu_{A_n}, S_{A_n}, \alpha_{A_n}$. By Lemma 1 and its corollaries we see that $\alpha_n \leqq S_n \alpha_n$, $\bigvee_{i=-\infty}^{\infty} S_n^i \alpha_n = \varepsilon_n$. If S_n is not ergodic for some n then there exists a $B \in \varepsilon_n$ with $0 < \mu(B) < \mu(A_n)$ and $S_n B = B$. By lemma 2 we have $B \in \alpha_n$ and hence $B \in \alpha$. Now $B = A_n \cap \bigcup_{k=1}^{\infty} T^{-k} B$ and if $C = \bigcup_{k=1}^{\infty} T^{-k} B$ then $TC = C$, $C \in \alpha$ and so $C \in \bigwedge_{i=-\infty}^{\infty} T^i \alpha = \nu$ i.e. $\mu(C) = 0$ or $\mu(X - C) = 0$. But $0 < \mu(B) \leqq \mu(C)$ and so we get $\mu(X - C) = 0$, which in turn gives $A_n \cap C = A_n$ i.e. $\mu(B) = \mu(A_n)$. This is a contradiction and so we deduce that S_n is, ergodic. The result then follows from lemma 3.

Bibliography

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(Received April 9, 1966.)