On minimal ideals in the circle composition semigroup of a ring

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Let R be a ring, and let $\circ$ denote the circle composition on R defined by $a\circ b= =a+b-ab$. In this note we discuss the existence of minimal ideals in $(R, \circ)$ and their significance in R. In § 1 we show that $(R, \circ)$ has a completely simple minimal ideal K if and only if R contains an idempotent e which is an identity for R modulo its Jacobson radical. If this is the case then the ideal $I(R)$ of the ring R generated by $K-K$ is seen to be a radical-like ideal which is zero if and only if R has an identity. A necessary and sufficient condition is given for R to be a splitting extension of $I(R)$ by a subring eRe for an idempotent e of R.

An interesting question is the determination of those simple rings R for which $(R, \odot)$ is simple. Saşiada's example ([5]) shows that R may be a radical ring. Our results imply that if $(R, \circ)$ is completely simple then R must be a radical ring. Example B below shows, however, that there are semi-simple non-simple rings for which $(R, \circ)$ is simple. By taking quotients one may easily construct from the ring in example B a ring R for which both $(R, \circ)$ and the multiplicative semigroup $(R, \cdot)$ are simple.

1. We denote by $J(R)$ the (Jacobson) radical of the ring R (see [4], ch. 1).

An idempotent e in R will be called *principal* if

$$(1-e)R+R(1-e)\subset J(R).$$

(Note that we use 1 only as a notational device. R may or may not have an identity). A principal idempotent is always principal in the classical sense (see [1], p. 25); however the converse does not hold. If $e=0$ is principal, then $J(R)=R$. If R has an identity 1, then 1 is the only principal idempotent of R. Our results become trivial in both these cases.

It is easy to see that a semi-primary SBI ring ([4], ch. III), and hence a ring with d.c.c., has a principal idempotent.

Let e be an idempotent. We define

$$P_e=(1-e)Re+eR(1-e)+(1-e)R(1-e).$$

P_e is the sum of the last three terms of the two-sided Peirce decomposition of R with respect to e. We note that $R=eRe+P_e$ is a direct sum decomposition qua abelian groups and that

$$P_e=(1-e)R+R(1-e).$$

Thus an idempotent e is principal if and only if $P_e \subset J(R)$.

We further define

$$G_e = e \circ J(R) \circ e = (1-e)J(R)(1-e)+e$$

$$L_e = J(R) \circ e \quad = J(R)(1-e)+e$$

$$R_e = e \circ J(R) \quad = (1-e)J(R)+e$$

$$K_e = L_e \circ R_e \quad = J(R) \circ e \circ J(R).$$

Lemma. 1. *$(G_e, \circ)$ is a group for any idempotent e in R.*

Proof. Since $e = e \circ e = e \circ 0 \circ e$, it follows that $e \in G_e$ and that e is an identity for G_e. Let now $g \in G_e$. Then $g = y + e$ where $y \in (1-e)J(R)(1-e) \subseteq J(R)$. Since $y \in J(R)$, y has a quasi-inverse z. Since $0 = y \circ z = y + z - yz$, it follows that $z = yz - y = (1-e)yz - (1-e)y = (1-e)(yz-y) = (1-e)z$. Similarly $z = z(1-e)$. Hence $z \in (1-e)\, J(R)(1-e)$ and $h = z + e$ is in G_e. Now, $ez = ze = ey = ye = 0$ and $z \circ y = y \circ z = 0$. Thus $h \circ g = g \circ h = e$, every element of G_e has an inverse, and the lemma is proved.

A semigroup is called *simple* if it contains no proper ideals, and *completely simple* if it is simple and every element lies in a subgroup, i.e. a subsemigroup which is a group (see [3]). We shall need the following result:

Theorem. (A. H. Clifford [2]). *If L and R are minimal left and minimal right ideals respectively of a semigroup S, then $K = LR$ is a minimal (two-sided) ideal of S and is a completely simple semigroup.*

This enables us to prove

Lemma 2. *If e is a principal idempotent of R, then K_e is a completely simple minimal ideal of $(R, \circ)$.*

Proof. Since $K_e = L_e \circ R_e$, it suffices, by Clifford's Theorem, to show that L_e and R_e are minimal left and right ideals. By symmetry it suffices to show that $L = L_e$ is a minimal left ideal.

Using the fact that e is principal, and thus that $R(1-e) = J(R)(1-e)$ we have that

$$R \circ L = R \circ J(R) \circ e \subset R \circ e = R(1-e)+e = J(R)(1-e)+e = L.$$

Thus L is a left ideal. Let now T be any left ideal of $(R, \circ)$ with $T \subset L$. Then $e \circ T \subset L \cap T \cap G_e$ since $e \circ L = G_e$. Therefore G_e meets T and, since G_e is a group, $G_e \subset T$. In particular, $e \in T$ and $L = J(R) \circ e \subset T$. Thus $T = L$ and L is indeed minimal.

Corollary 3. If e and f are any two principal idempotents, then $K_e = K_f$.

Corollary 4. If e is a principal idempotent, then

$$G_e = e \circ R \circ e$$

$$L_e = R \circ e$$

$$R_e = e \circ R$$

$$K_e = R \circ e \circ R.$$

PROOF. Since $e \in L_e$, $R \circ e \subset L_e$. But L_e is minimal. Thus $R \circ e = L_e$. The other parts follow similarly.

Theorem 1. *If R is a ring, $(R, \circ)$ has a completely simple minimal ideal K if and only if R contains a principal idempotent. Further, an idempotent is in K if and only if it is principal.*

PROOF. The sufficiency was proved in Lemma 2 and Corollary 3. To show the necessity of the conditions, assume that K is a completely simple minimal ideal of $(R, \circ)$ and let e be any idempotent of K. We show that e is principal. First we note that $(1-e)R = e \circ R - e$. Let then $x = e \circ r - e$ for some $r \in R$. Since K is completely simple $e \circ r \circ e$ lies in the subgroup $e \circ K \circ e = e \circ R \circ e$ of K containing e (see [3], Lemma 2. 46, p. 77). There then exists $s \in R$ with

$$e = (e \circ s \circ e) \circ (e \circ r \circ e) = e \circ (s \circ e \circ r) \circ e. \tag{1}$$

Let $z = e \circ s \circ e$ and $y = zx - x$. This yields $zx = (1-e)s(1-e)r$. Thus $y \in (1-e)R$. If we can show that $y \circ x = 0$, then it will follow that $(1-e)R$ is a quasi regular right ideal and therefore is contained in $J(R)$, ([4], ch. 1). We now calculate:

$$\begin{aligned} y \circ x &= zx - zx^2 + x^2 \\ &= (1-e)(s - sr + ser + r)(1-e)r \\ &= (1-e)(s + e - se + r - sr - er + ser)(1-e)r \\ &= (1-e)(s \circ e \circ r)(1-e)r. \end{aligned}$$

Now, by (1) we have

$$e = e \circ (s \circ e \circ r) \circ e = (1-e)(s \circ e \circ r)(1-e) + e$$

which implies that $(1-e)(s \circ e \circ r)(1-e) = 0$ and $y \circ x = 0$. Similarly, $R(1-e) \subset J(R)$. Thus e is principal. Since e was any idempotent of K, the theorem is proved.

2. In this section, R is a ring with a principal idempotent e and K is the minimal ideal of $(R, \circ)$.

We denote by $I(R)$ the ideal of R generated by $K - e$. $I(R)$ does not depend on the choice of e since $K - a$ and $K - b$ generate the same ideal whenever a and b are both elements of K. Since $K \subset J(R) + e$, it follows immediately that $I(R) \subset J(R)$.

Theorem 2. *If R contains a principal idempotent, then*

(i) $I(R) = 0$ *if and only if R has an identity*

and

(ii) $I(R/I(R)) = 0$.

PROOF. (i) If $I(R) = 0$, then $K - e = 0$. Since $R \circ e \subset K$ we must also have that $R \circ e - e = 0$. This implies that e is a right identity for R. Likewise, e is a left identity. Conversely, if e is an identity for R, then $K = \{e\}$ and $I(R) = 0$. The verification of (ii) is routine and will be omitted.

Since $I(R)\varphi \subset I(R\varphi)$ for any homomorphism φ of R, we easily obtain

Corollary 5. $I(R)$ is the intersection of all ideals A of R for which R/A has an identity.

We can now connect $I(R)$ and the Peirce decomposition of R:

Lemma. 6. *$I(R)$ is the subring generated by P_e.*

PROOF. Since $R=eRe+P_e$, the factor by the ideal generated by P_e has an identity and thus this ideal contains $I(R)$. However, $(1-e)R\cup R(1-e)\subset(L_e-e)\cup(R_e-e)\subset K-e$. Since $P_e=(1-e)R+R(1-e)$ this implies that $P_e\subset I(R)$. To complete the proof of the lemma, we need only observe that the subring and the ideal generated by P_e coincide.

Corollary 7. If either $I(R)^2=0$ or R contains a principal central idempotent, then $R=eRe+I(R)$ and $eRe\cap I(R)=0$.

PROOF. One need only notice that in both these cases P_e is already a subring, and thus that $I(R)=P_e$.

By a *linear variety* in a ring R we mean a translate $a+M$ of a subgroup M of the additive group of R. It is easily verified that V is a linear variety of R if and only if all finite sums $\sum n_i v_i$ lie in V whenever the $v_i\in V$ and the n_i are integers whose sum is 1.

It is easy to verify that if T is an ideal of $(R, \circ)$ the linear variety generated by T is also an ideal of $(R, \circ)$ and that if T is both an ideal of $(R,\circ)$ and a linear variety then, for any $t\in T$, $T-t$ is an ideal of R. However all the ideals of R may not be obtainable in this fashion.

Theorem 3. *Let e be a principal idempotent of R. Then, K is a linear variety of R if and only if $R=eRe+I(R)$ (a direct sum qua abelian groups). When this occurs, then $K=P_e+e$ and $I(R)=P_e$.*

PROOF. We first assume that K is a linear variety. We need only show that $I(R)=P_e$. To this effect, we first show that $eKe=e$: Let $x\in K$. Since K is a linear variety, as well as an ideal of $(R, \circ)$ containing e, we have that

$$exe=x-x\circ e-e\circ x+e\circ x\circ e+e$$

lies in K. Since K is a union of groups by Theorem 1, there exists an idempotent $g\in K$ for which $g\circ exe=exe=exe\circ g$. Thus $g=gexe$ and $g=exeg$ and further $ge=eg=g$. Whence $e\circ g=e=g\circ e$. Therefore $e=g\circ e\circ g\in g\circ R\circ g$ which is a group by lemma 1 and corollary 4. Since a group contains a single idempotent, e must equal g. This shows that $exe=e$ for all $x\in K$.

Now, by a previous remark, since K is a linear variety, $K-e$ is an ideal of R and therefore $K-e=I(R)$. Thus $eI(R)e=e(K-e)e=eKe-e=e-e=0$ and it follows easily that $I(R)\subset P_e$. Lemma 6 then forces P_e to coincide with $I(R)$.

Assume now, to prove the converse, that $R=eRe+I(R)$ is a direct sum qua abelian groups. Since $R=eRe+P_e$ is such a direct sum and $P_e\subseteqq I(R)$, it follows that $P_e=I(R)$. Recalling that $K-e$ generates $I(R)=P_e$, we see that $K-e\subseteqq P_e$. To prove the opposite inclusion we show first that any idempotent $y+e\in P_e+e$ must lie in K:

$(e+y)^2=e+y$ and hence $ey+ye+y^2=y$. Multiplying by e on the left yields $eye+ey^2=0$ and hence $(e+ey)\circ(e+y)=(e+y)$. To show that $e+y\in K$, it therefore suffices to show that $e+ey\in K$. However this is clear since $e+eP_e=e+eR(1-e)\subset L_e\subset K$.

Let now $e+a$ be an arbitrary element of $e+P_e$. Since $a\in P_e=I(R)$ we have that $a-2ea\in P_e$. Let q be the quasi-universe of $a-2ea$ in P_e. (Such a q exists since P_e is an ideal contained in $J(R)$). Let $y=eq+qe+qeq$. Since $eP_ee=0$, it follows that $y+e$ is an idempotent in P_e+e and thus that $y+e\in K$. If we can show that $(y+e)\circ(a+e)=a+e$ we will have shown that $a+e\in K$ and thus that $K=P_e+e$, which will prove that K is a linear variety. Now,

$$(y+e)\circ(a+e) = a+e+y-ya-ye-ea.$$

Substituting $eq+qe+qeq$ for y and using the fact that $eqe=0$, we find that

$$(y+e)\circ(a+e) = a+e+(1+q)(eq-ea-eqa).$$

It therefore suffices to show that $eq-ea-eqa = 0$. However,

$$0=e0=e\big(q\circ(a-2ea)\big)=e(q+a-2ea-qa+2qea)=eq-ea-eqa.$$

This then completes the proof of Theorem 3.

3. *Examples.*

A. The ring of all real 3×3 matrices of the form

$$\begin{pmatrix} x & 0 & 0 \\ y & 0 & 0 \\ z & r & s \end{pmatrix}$$

provides an example of a ring R where K is not a linear variety. This is easily verified since K consists of all matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ -xy & y & 0 \end{pmatrix}$$

B. We now give an example of a ring R for which $K=R$ is simple but not completely simple:

Let L be the ring of all linear trnsformations on a vector space V of uncountable dimension, and let R be the subring of L consisting of all linear transformations σ such that $\dim V\sigma<\dim V$.

We first note that any ideal I of $(R, \circ)$ must contain an idempotent, for if $\sigma\in I$ and e is any projection on $V\sigma$, then $e=\sigma\circ e\in I$. To prove that $(R, \circ)$ is simple we need only show that any ideal of $(R, \circ)$ contains 0, and we achieve this by exhibiting, for any idempotent $e\in R$, σ and τ in R for which $\sigma\circ e\circ\tau=0$. Let then $e\in R$ be the projection on the subspace M_1 along the subspace W. Since $\dim W>\dim M_1$, we can find a sequence of subspaces $M_2, M_3, M_4, \ldots$, all isomorphic to M_1 such that

$$W=(M_2\oplus M_3\oplus \ldots)\oplus U.$$

We choose bases $\{m_\alpha^i\colon i=1, 2, \ldots; \alpha\in\Lambda\}$ for the M_i and we define σ and τ by

$$m_\alpha^1\sigma = m_\alpha^2;\quad m_\alpha^i\sigma = m_\alpha^i+m_\alpha^{i+1} \qquad i>1,\quad U\sigma=0$$
$$m_\alpha^1\tau = m_\alpha^1;\quad m_\alpha^i\tau = m_\alpha^i+m_\alpha^{i-1} \qquad i>1,\quad U\tau=0$$

Then it is clear that both σ and τ are in R, and one verifies that $\sigma \circ e \circ \tau = 0$. $(R, \circ)$ is then simple.

If $(R, \circ)$ were completely simple, then, by Theorem 1, all idempotents of R would be principal. But it is clear that R has no principal idempotents.

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