

On four extensions of the functional equation $|f(x+iy)| = |f(x)+f(iy)|$

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§ 1. Introduction

R. M. ROBINSON solved the following functional equation:

$$(1) \quad |f(x+iy)| = |f(x)+f(iy)|,$$

where $f(z)$ is regular for $|z| < r$ and x, y are real. (See [1].)

In this paper we shall solve four extensions of (1).

Firstly we shall solve the following three functional equations which were presented by J. ACZÉL:

$$(2) \quad |f(x+iy)| = |f(x)+af(iy)|,$$

$$(3) \quad |f(x+iy)| = |af(x)+f(iy)|,$$

$$(4) \quad |f(x+iy)| = |af(x)+bf(iy)|,$$

where $f(z)$ is an entire function of z , and x, y are real, and a, b are complex constants. Here, putting $a=1, b=1$, we have (1), and putting $a=1, b=-1$, we have $|f(x+iy)| = |f(x)-f(iy)|$ which was solved in [2]. We shall reduce (4) to (2), (3).

Secondly we shall solve the following functional equation which has a geometric meaning:

$$(5) \quad |f(x+iy)+f(0)| = |f(x)+f(iy)|,$$

where $f(z)$ is an entire function of z , and x, y are real. Here, putting $f(0)=0$, we have (1).

§ 2. On the functional equations (2), (3), (4)

Theorem 1. *If $f(z)$ is an entire function of z and satisfies the functional equation (2) for real values of x and y , then the solutions of (2) are the following and only these:*

Case (i) $a=0$. $f(z) = C \exp(\alpha z)$,

where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (ii) $a=1$. $f(z) = Cz$, or $f(z) = C \sin \alpha z$, or $f(z) = C \sin h\alpha z$, where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (iii) $a=-1$. $f(z) = Az + Bz^2$, or $f(z) = A \sin \alpha z + B \cos \alpha z - B$, or $f(z) = A \sin h\alpha z + B \cos h\alpha z - B$,

where A, B are arbitrary complex constants and α is an arbitrary real constant.

Case (iv) other than the cases (i), (ii), (iii). $f(z) \equiv 0$ when $|1+a| \neq 1$, $f(z) \equiv \text{arbitrary const.}$ when $|1+a|=1$.

PROOF. Case (A): $f(0) \neq 0$. Putting $x=0, y=0$ in (2), by $f(0) \neq 0$ we have

$$(6) \quad |1+a|=1.$$

We may assume that $f(z) \neq \text{const.}$ Putting $g(z) = \frac{f(z)}{f(0)}$, we have

$$g(z) = 1 + \sum_{n=0}^{+\infty} a_{p+n} z^{p+n}$$

($a_p \neq 0$, where p is a natural number).

By (2) we have

$$(7) \quad g(x+iy)\overline{g(x+iy)} = (g(x) + ag(iy))(\overline{g(x) + ag(iy)}).$$

Equating the coefficients of x^p of both sides in (7), we have

$$(8) \quad \bar{a}a_p + a\bar{a}_p = 0.$$

Now, we shall prove that $a=0$. Suppose that $p>1$. Then, equating the coefficients of $x^{p-1}y$ of both sides in (7), we have

$$(9) \quad \bar{a}_p = a_p.$$

Since $a_p \neq 0$, by (8), (9) a is purely imaginary. Hence, by (6) we have $a=0$. Next, suppose that $p=1$. Equating the coefficients of x, iy of both sides in (7), we have

$$(10) \quad \bar{a}a_1 + a\bar{a}_1 = 0,$$

$$(11) \quad a_1 - \bar{a}_1 = a(1+\bar{a})a_1 - \bar{a}(1+a)\bar{a}_1.$$

By (6) we have

$$(12) \quad a = e^{i\theta} - 1,$$

where θ is a real constant. Substituting (12) in (10), we have

$$e^{i\theta} = 1 \quad \text{or} \quad a_1 = e^{i\theta}\bar{a}_1.$$

When $e^{i\theta} = 1$, by (12) we have $a=0$. When $a_1 = e^{i\theta}\bar{a}_1$, by (11), (12) we have

$$e^{i\theta}\bar{a}_1 - \bar{a}_1 = (e^{i\theta} - 1)e^{-i\theta}e^{i\theta}\bar{a}_1 - (e^{-i\theta} - 1)e^{i\theta}\bar{a}_1.$$

Since $\bar{a}_1 \neq 0$, we have $e^{i\theta} = 1$. Hence, by (12) we have $a=0$.

Thus, by (2) we have

$$(13) \quad |f(x+iy)|^2 = |f(x)|^2.$$

Putting $f(x+iy) = u + iv$ where u, v are real, by (13) we have $\frac{\partial}{\partial y}(u^2 + v^2) = 0$ in $|z| < +\infty$. Hence we have $uu_y + vv_y = 0$ in $|z| < +\infty$. Hence, by the Cauchy—Riemann equations we have in $|z| < +\infty$

$$(14) \quad uv_x - vu_x = 0.$$

By our assumption $f(z) \neq 0$. Choosing a vicinity V properly, we have $f(z) \neq 0$ in V . Hence we have in V

$$\frac{f'(z)}{f(z)} = \frac{u_x + iv_x}{u + iv} = \frac{uu_x + vv_x}{u^2 + v^2} + i \frac{uv_x - vu_x}{u^2 + v^2},$$

where $z = x + iy$ (x, y real). Hence, by (14) we have $\operatorname{Im} \left(\frac{f'(z)}{f(z)} \right) = 0$ in V . Hence

we have $\frac{f'(z)}{f(z)} = \alpha$ where α is a real constant. Solving this differential equation, we have $f(z) = C \exp(\alpha z)$ where C is a complex constant.

Case (B): $f(0) = 0$. By (2) we have

$$(15) \quad f(x+iy)\overline{f(x+iy)} = (f(x) + af(iy))(\overline{f(x) + af(iy)}).$$

We may assume that $f(z) \neq \text{const.}$ Using the power series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ ($a_n \neq 0$ where n is a natural number) and equating the terms of degree $2p$ with respect to x and y of both sides in (15), we have $p = 1$.

Putting $g(z) = \frac{f(z)}{a_1}$, we have in $|z| < +\infty$

$$g(z) = z + b_2 z^2 + b_3 z^3 + \dots + b_n z^n + \dots$$

By (2) we have

$$(16) \quad g(x+iy)\overline{g(x+iy)} = (g(x) + ag(iy))(\overline{g(x) + ag(iy)}).$$

Equating the coefficients of xy and y^2 , we have

$$(17) \quad \bar{a} = a,$$

$$(18) \quad |a| = 1.$$

By (17), (18) we have $a = 1$ or $a = -1$. By [1] and the previous paper [2] the theorem is proved.

Theorem 2. If $f(z)$ is an entire function of z and satisfies the functional equation (3) for real values of x and y , then the solutions of (3) are the following and only these:

Case (i) $a = 0$. $f(z) = C \exp(\alpha z)$,

where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (ii) $a = 1$. $f(z) = Cz$, or $f(z) = C \sin \alpha z$, or $f(z) = C \sin h\alpha z$,

where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (iii) $a = -1$. $f(z) = Az + Bz^2$ or $f(z) = A \sin \alpha z + B \cos \alpha z - B$, or $f(z) = A \sin h\alpha z + B \cos h\alpha z - B$,

where A, B are arbitrary complex constants and α is an arbitrary real constant.

Case (iv) $a \neq 0, 1, -1$. $f(z) \equiv 0$ when $|1+a| \neq 1$, $f(z) \equiv \text{arbitrary const.}$ when $|1+a| = 1$.

PROOF. Putting $g(z) = \overline{f(i\bar{z})}$, $g(z)$ is an entire function of z and by (3) we have

$$|g(x+iy)| = |g(x) + \bar{a}g(iy)|,$$

where x, y are real. Hence, by Theorem 1 the theorem is proved.

Theorem 3. *If $f(z)$ is an entire function of z and satisfies the functional equation (4) for real values of x and y , then the solutions of (4) are the following and only these:*

Case (i) $|a|=1, b=0$. $f(z)=C \exp(\alpha z)$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (ii) $a=0, |b|=1$. $f(z)=C \exp(i\alpha z)$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (iii) $|a|=1, a=b$. $f(z)=Cz$, or $f(z)=C \sin \alpha z$, or $f(z)=C \sin h\alpha z$,
where C is an arbitrary complex constant and α is an arbitrary real constant.

Case (iv) $|a|=1, a=-b$. $f(z)=Az+Bz^2$, or $f(z)=A \sin \alpha z+B \cos \alpha z-B$,
or $f(z)=A \sin h\alpha z+B \cos h\alpha z-B$,
where A, B are arbitrary complex constants and α is an arbitrary real constant.

Case (v) other than the cases (i), (ii), (iii), (iv). $f(z)\equiv 0$ when $|a+b|\neq 1$,
 $f(z)\equiv$ arbitrary const. when $|a+b|=1$.

PROOF. Case (A) $|a|<1, |b|<1$. By (4) we have in $|x|<+\infty$

$$(19) \quad |f(x)| \leq |a| |f(x)| + |b| |f(0)|.$$

Since $|a|<1$, by (19) we have in $|x|<+\infty$

$$(20) \quad |f(x)| \leq \frac{|b| |f(0)|}{1-|a|}.$$

By (4) we have in $|y|<+\infty$

$$(21) \quad |f(iy)| \leq |a| |f(0)| + |b| |f(iy)|.$$

Since $|b|<1$, by (21) we have in $|y|<+\infty$

$$(22) \quad |f(iy)| \leq \frac{|a| |f(0)|}{1-|b|}.$$

By (4), (20), (22) we have in $|x+iy|<+\infty$

$$(23) \quad |f(x+iy)| \leq \frac{|a| |b| |f(0)|}{1-|a|} + \frac{|a| |b| |f(0)|}{1-|b|}.$$

By (23) and Liouville's theorem we have $f(z)\equiv \text{const.}$

Case (B) $|a|<1, |b|>1$.

Since $|a|<1$, by (20) we have in $|x|<+\infty$

$$(24) \quad |f(x)| \leq \frac{|b| |f(0)|}{1-|a|}.$$

By (4) we have in $|x+iy|<+\infty$ $|f(x+iy)| \leq |b| |f(iy)| - |a| |f(x)|$.

Hence we have in $|y|<+\infty$

$$(25) \quad |f(iy)| \leq \frac{|a| |f(0)|}{|b|-1}.$$

By (4), (24), (25) we have in $|x+iy| < +\infty$

$$(26) \quad |f(x+iy)| \leq \frac{|a||b||f(0)|}{1-|a|} + \frac{|b||a||f(0)|}{|b|-1}.$$

By (26) and Liouville's theorem we have $f(z) \equiv \text{const.}$

Case (C) $|a| > 1, |b| < 1$. Putting $g(z) = f(i\bar{z})$, $g(z)$ is an entire function of z and by (4) we have $|g(x+iy)| = |\bar{b}g(x) + \bar{a}g(iy)|$, where x, y are real. Hence, by the result of Case (B) we have $f(z) \equiv \text{const.}$

Case (D) $|a| > 1, |b| > 1$.
By (4) we have in $|x+iy| < +\infty$

$$|f(x+iy)| \leq |a||f(x)| - |b||f(iy)|.$$

Hence we have in $|x| < +\infty$

$$(27) \quad |f(x)| \leq \frac{|b||f(0)|}{|a|-1}.$$

Since $|b| > 1$, by (25) we have in $|y| < +\infty$

$$(28) \quad |f(iy)| \leq \frac{|a||f(0)|}{|b|-1}.$$

By (4), (27), (28) we have in $|x+iy| < +\infty$

$$(29) \quad |f(x+iy)| \leq \frac{|a||b||f(0)|}{|a|-1} + \frac{|a||b||f(0)|}{|b|-1}.$$

By (29) and Liouville's theorem we have $f(z) \equiv \text{const.}$

Case (E) other than the cases (A), (B), (C), (D).

When $|a| = 1$, by (4) we have

$$|f(x+iy)| = \left| f(x) + \frac{b}{a} f(iy) \right|.$$

Thus the solution of (4) reduces to that of (2). Next, when $|b| = 1$, by (4) we have

$$|f(x+iy)| = \left| \frac{a}{b} f(x) + f(iy) \right|.$$

Thus the solution of (4) reduces to that of (3). Thus the theorem is proved.

§ 3. On the functional equation (5)

Theorem 4. *If $f(z)$ is an entire function of z and satisfies the functional equation (5) for real values of x and y , then the solutions of (5) are the following and only these:*

$$f(z) = Az + B, \quad \text{or} \quad f(z) = A \sin \alpha z + B \cos \alpha z, \quad \text{or} \quad f(z) = A \sin h\alpha z + B \cos h\alpha z,$$

where A, B are arbitrary complex constants and α is an arbitrary real constant.

PROOF. We may assume that $f(0) \neq 0$. Using the power series $f(z) = \sum_{n=0}^{+\infty} a_n z^n$ and putting $g(z) = \frac{f(z)}{a_0}$ in $|z| < +\infty$, we have in $|z| < +\infty$

$$(30) \quad g(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \dots + b_n z^n + \dots$$

By (5) we have in $|z| < +\infty$

$$(31) \quad (g(z) + 1)(\overline{g(z)} + 1) = \left(g\left(\frac{z + \bar{z}}{2}\right) + g\left(\frac{z - \bar{z}}{2}\right) \right) \left(\overline{g\left(\frac{z + \bar{z}}{2}\right)} + \overline{g\left(\frac{z - \bar{z}}{2}\right)} \right).$$

Substituting (30) in (31) and equating the coefficients of z^2 of both sides, we have $2b_2 = b_2 + \bar{b}_2$. Hence we have $\bar{b}_2 = b_2$. Hence b_2 is real. Substituting (30) in (31) and equating the coefficients of z^n of both sides for $n > 2$, we have

$$\frac{2^{n-1} - 1}{2^{n-1}} b_n - \frac{1}{2^{n-1}} \bar{b}_n = P(b_1, b_2, b_3, \dots, b_{n-1}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-1}),$$

where $n(>2)$ is even and P is a polynomial in the earlier coefficients $b_1, b_2, b_3, \dots, \dots, b_{n-1}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-1}$, and

$$\frac{2^{n-1} - 1}{2^{n-1}} b_n = P(b_1, b_2, b_3, \dots, b_{n-1}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-1}),$$

where $n(>2)$ is odd, and P is a polynomial in the earlier coefficients $b_1, b_2, b_3, \dots, \dots, b_{n-1}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \dots, \bar{b}_{n-1}$.

Since $2^{n-1} - 2 \neq 0$ (>0) for $n > 2$, the remaining coefficients b_n ($n > 2$) are uniquely determined in terms of b_1, b_2 where b_2 is real. On the other hand

$$g(z) = \frac{b_1}{\sqrt{-2b_2}} \sin \sqrt{-2b_2} z + \cos \sqrt{-2b_2} z = 1 + b_1 z + b_2 z^2 + \dots,$$

$$\text{or} \quad g(z) = \frac{b_1}{\sqrt{2b_2}} \sinh \sqrt{2b_2} z + \cosh \sqrt{2b_2} z = 1 + b_1 z + b_2 z^2 + \dots,$$

$$\text{or} \quad g(z) = 1 + b_1 z,$$

respectively, are solutions of the functional equation $|g(x + iy) + 1| = |g(x) + g(iy)|$, if b_2 is negative or positive or 0.

Since the remaining coefficients b_n ($n > 2$) are uniquely determined in terms of b_1, b_2 , there can be no other normalized solutions. Thus the theorem is proved.

Example. (See [3].) By the above theorem we can solve the following functional equation under the hypothesis that $f(x)$ is an entire function of x (x complex):

$$(32) \quad |f(x + y) + f(x - y)| = |f(x + \bar{y}) + f(x - \bar{y})|,$$

where x, y are complex.

Solution. Putting $x = y = \frac{s + it}{2}$ in (32) where s, t are real, we have

$$(33) \quad |f(s + it) + f(0)| = |f(s) + f(it)|.$$

By the above theorem the solutions of (32) are the following and only these:

$$f(z) = A + Bz, \text{ or } f(z) = A \sin \alpha z + B \cos \alpha z, \text{ or } f(z) = A \sin h\alpha z + B \cos h\alpha z,$$

where A, B are arbitrary complex constants and α is an arbitrary real constant.

Remark. The sufficiency of this example gives the following theorem:

For a family of confocal ellipses and hyperbolas, let M, N be the middle points of the two diagonals of a curvilinear rectangle formed by any two ellipses and two hyperbolas and let O be the center of this family. Then we have $\overline{OM} = \overline{ON}$ (and the above statement on (32) gives all curves which have this property).

References

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