

A three-term relation for the Dedekind-Rademacher sums

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1. For real x put

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & (x \neq \text{integer}) \\ 0 & (x = \text{integer}) \end{cases}$$

and define the Dedekind sum

$$s(b, a) = \sum_{r(\bmod a)} \left(\left(\frac{r}{a} \right) \right) \left(\left(\frac{br}{a} \right) \right).$$

RADEMACHER ([3]) has proved the following three-term relation satisfied by $s(b, a)$:

$$(1.1) \quad s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \right),$$

where

$$(a, b) = (b, c) = (c, a) = 1$$

and a', b', c' are defined by

$$aa' \equiv 1 \pmod{bc}, \quad bb' \equiv 1 \pmod{ca}, \quad cc' \equiv 1 \pmod{ab}.$$

In particular, when $c = c' = 1$, (1.1) reduces to the familiar reciprocity formula

$$(1.2) \quad s(b, a) + s(a, b) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right).$$

In a more recent paper [4], Rademacher has introduced the sum

$$(1.3) \quad s(h, k; x, y) = \sum_{r(\bmod k)} \left(\left(h \frac{r+y}{k} + x \right) \right) \left(\left(\frac{r+y}{k} \right) \right)$$

and proved the reciprocity formula

$$(1.4) \quad s(h, k; x, y) + s(k, h; y, x) = ((x))((y)) + \frac{1}{2} \left\{ \frac{h}{k} \Psi_2(y) + \frac{1}{hk} \Psi_2(hy + kx) + \frac{k}{h} \Psi_2(x) \right\},$$

where $(h, k) = 1$, x and y are not both integers and $\Psi_2(x) = B_2(x - [x])$, where

$$B_2(x) = x^2 - x + \frac{1}{6},$$

the Bernoulli polynomial of degree 2. The writer [1], [2] has proved a generalization of (1.4).

In the present paper we obtain a three-term relation satisfied by $s(h, k; x, y)$. It will however be convenient to change the notation defined above. To begin with, we put

$$(1.5) \quad \Phi(x) = x - [x] - \frac{1}{2}$$

for all real x . In the next place we define

$$(1.6) \quad s(a, b, c; x, y, z) = \sum_{t(\bmod c)} \Phi\left(a \frac{t+z}{c} - x\right) \Phi\left(y - b \frac{t+z}{c}\right).$$

Despite the presence of the additional parameters, $s(a, b, c; x, y, z)$ is really no more general than $s(h, k; x, y)$ as defined by (1.3).

We shall prove the following

Theorem. Let $(a, b) = (b, c) = (c, a) = 1$. Then we have

$$(1.7) \quad s(a, b, c; x, y, z) + s(b, c, a; y, z, x) + s(c, a, b; z, x, y) = \\ = \delta - \frac{a}{2bc} \Psi_2(cy - bz) - \frac{b}{2ca} \Psi_2(az - cx) - \frac{c}{2ab} \Psi_2(bx - ay),$$

where $\delta = 1$ if integers r, s, t exist such that

$$(1.8) \quad \frac{r+x}{a} = \frac{s+y}{b} = \frac{t+z}{c};$$

$\delta = 0$ otherwise.

2. We shall need a few preliminary results. Clearly $\Phi(x+1) = \Phi(x)$; also it is familiar that

$$(2.1) \quad \Phi(-x) = -\Phi(x),$$

provided x is not an integer. We recall also that

$$(2.2) \quad \sum_{r(\bmod k)} \Phi\left(x + \frac{r}{k}\right) = \Phi(kx).$$

Applying (2.2) to (1.6) we get

$$(2.3) \quad s(a, b, c; x, y, z) = \sum_{r, s, t} \Phi\left(\frac{t+z}{c} - \frac{r+x}{a}\right) \Phi\left(\frac{s+y}{b} - \frac{t+z}{c}\right),$$

where r, s, t run through complete residue systems, modulo a, b, c respectively. If we put

$$(2.4) \quad \xi = \frac{r+x}{a}, \quad \eta = \frac{s+y}{b}, \quad \zeta = \frac{t+z}{c},$$

we may rewrite (2.3) compactly as

$$(2.5) \quad s(a, b, c; x, y, z) = \sum_{r, s, t} \Phi(\zeta - \xi) \Phi(\eta - \zeta).$$

The following lemmas will be used later.

Lemma 1. *We have*

$$(2.6) \quad \sum_{r(\bmod a)} \Phi^2 \left(\frac{r+x}{a} \right) = \frac{1}{a} \Psi_2(x) + \frac{1}{12} a.$$

PROOF. We may assume, without loss of generality, that $0 \leq x < 1$. Then

$$\sum_{r(\bmod a)} \Phi^2 \left(\frac{r+x}{a} \right) = \sum_{r=0}^{a-1} \left(\frac{r+x}{a} - \frac{1}{2} \right)^2 = \frac{1}{a^2} \sum_{r=0}^{a-1} \left(r+x-\frac{a}{2} \right)^2$$

We recall that

$$\sum_{r=0}^{a-1} (r+y)^2 = \frac{1}{3} \{B_3(y+a) - B_3(y)\},$$

where

$$B_3(y) = y^3 - \frac{3}{2} y^2 + \frac{1}{2} y.$$

Thus

$$\sum_{r(\bmod a)} \Phi^2 \left(\frac{r+x}{a} \right) = \frac{1}{3a^2} \left\{ B_3 \left(x + \frac{1}{2} a \right) - B_3 \left(x - \frac{1}{2} a \right) \right\},$$

which reduces to (2.6).

Lemma 2. *Let $(a, b) = 1$. Then*

$$(2.7) \quad \sum_{\substack{r(\bmod a) \\ s(\bmod b)}} \Phi^2 \left(\frac{r+x}{a} - \frac{s+y}{b} \right) = \frac{1}{ab} \Psi_2(bx-ay) + \frac{1}{12} ab.$$

Since

$$\Phi \left(\frac{r+x}{a} - \frac{s+y}{b} \right) = \Phi \left(\frac{br-as}{ab} + \frac{bx-ay}{ab} \right),$$

we have

$$\sum_{r,s} \Phi^2 \left(\frac{r+x}{a} - \frac{s+y}{b} \right) = \sum_{t(\bmod ab)} \Phi^2 \left(\frac{t}{ab} + \frac{bx-ay}{ab} \right)$$

and (2.7) follows at once from (2.5).

3. We shall now prove the theorem stated in § 1. Let S denote the left hand side of (1.7). Then by (2.5) we have

$$(3.1) \quad S = \sum_{r,s,t} \{ \Phi(\xi-\eta) \Phi(\eta-\zeta) + \Phi(\eta-\zeta) \Phi(\zeta-\xi) + \Phi(\zeta-\xi) \Phi(\xi-\eta) \},$$

where ξ, η, ζ are defined by (2.4). Now consider the sum

$$(3.2) \quad T = \sum_{r,s,t} \{ \Phi(\xi-\eta) + \Phi(\eta-\zeta) + \Phi(\zeta-\xi) \}^2.$$

In view of (1.5) we have

$$(3.3) \quad T = \sum_{r,s,t} \{ [\xi-\eta] + [\eta-\zeta] + [\zeta-\xi] + \frac{3}{2} \}^2.$$

Clearly there is no loss in generality in assuming that

$$(3.4) \quad 0 \leq x < 1, \quad 0 \leq y < 1, \quad 0 \leq z < 1$$

and that

$$(3.5) \quad 0 \leq r < a, \quad 0 \leq s < b, \quad 0 \leq t < c.$$

It follows from (3.4) and (3.5) that

$$0 \leq \xi < 1, \quad 0 \leq \eta < 1, \quad 0 \leq \zeta < 1$$

and therefore

$$|\xi - \eta| < 1, \quad |\eta - \zeta| < 1, \quad |\zeta - \xi| < 1.$$

Consequently each of $[\xi - \eta]$, $[\eta - \zeta]$, $[\zeta - \xi]$ is equal to 0 or -1 .

Two possibilities must be considered:

Case I. Integers r, s, t exist such that

$$(3.6) \quad \frac{r+x}{a} = \frac{s+y}{b} = \frac{t+z}{c}.$$

If such integers exist they are uniquely determined. For assume a second triple r', s', t' such that

$$\frac{r'+x}{a} = \frac{s'+y}{b} = \frac{t'+z}{c}.$$

Then clearly

$$\frac{r-r'}{a} = \frac{s-s'}{b} = \frac{t-t'}{c},$$

which implies

$$r \equiv r' \pmod{a}, \quad s \equiv s' \pmod{b}, \quad t \equiv t' \pmod{c}.$$

Case II. (3.6) is never satisfied.

If r, s, t satisfy (3.6) it is evident that

$$[\xi - \eta] + [\eta - \zeta] + [\zeta - \xi] = 0.$$

For all other triples, however, we have

$$[\xi - \eta] + [\eta - \zeta] + [\zeta - \xi] = -1 \quad \text{or} \quad -2.$$

It therefore follows from (3.3) that

$$(3.7) \quad T = \begin{cases} \frac{1}{4}abc + 2 & \text{(case I)} \\ \frac{1}{4}abc & \text{(case II)}. \end{cases}$$

Now, on the other hand, it is clear from (3.1) and (3.2) that

$$(3.8) \quad \begin{aligned} T &= 2S + \sum_{r,s,t} \{ \Phi^2(\xi - \eta) + \Phi^2(\eta - \zeta) + \Phi^2(\zeta - \xi) \} \\ &= 2S + a \sum_{s,t} \Phi^2(\eta - \zeta) + b \sum_{t,r} \Phi^2(\zeta - \xi) + c \sum_{r,s} \Phi^2(\xi - \eta). \end{aligned}$$

Applying Lemma 2, we get

$$(3.9) \quad S = \frac{1}{2}T - \frac{abc}{8} - \frac{a}{2bc} \Psi_2(cy - bz) - \frac{b}{2ca} \Psi_2(az - cx) - \frac{c}{2ab} \Psi_2(bx - ay).$$

If we put

$$\delta = \begin{cases} 1 & (\text{case I}) \\ 0 & (\text{case II}), \end{cases}$$

then by (3. 7)

$$\frac{1}{2}T - \frac{abc}{8} = \delta$$

and (3. 9) reduces to (1. 7). This completes the proof of the theorem.

4. We assume in what follows that $0 \leq x < 1$, $0 \leq y < 1$, $0 \leq z < 1$. When $x=y=z=0$, we have

$$\begin{aligned} s(a, b, c; 0, 0, 0) &= \sum_{t(\bmod c)} \Phi\left(\frac{at}{c}\right) \Phi\left(-\frac{bt}{c}\right) \\ &= \frac{1}{4} - \sum_{t(\bmod c)} \left(\left(\frac{at}{c}\right)\right) \left(\left(\frac{bt}{c}\right)\right) \\ &= \frac{1}{4} - \sum_{t(\bmod c)} \left(\left(\frac{ab't}{c}\right)\right) \left(\left(\frac{t}{c}\right)\right), \end{aligned}$$

so that

$$s(a, b, c; 0, 0, 0) = \frac{1}{4} - s(ab', c).$$

Thus (1. 7) becomes

$$s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} \right),$$

in agreement with (1. 1).

In the next place, if we take $c=1$, $z=0$ and replace y by $-y$, (1. 6) implies

$$s(a, b, 1; x, -y, 0) = \Phi(-x)\Phi(-y),$$

$$s(b, 1, a; -y, 0, x) = \sum_{r(\bmod a)} \Phi\left(b\frac{r+x}{a} + y\right) \Phi\left(-\frac{r+x}{a}\right),$$

$$s(1, a, b; 0, x, -y) = \sum_{s(\bmod b)} \Phi\left(-\frac{s+y}{b}\right) \Phi\left(x + a\frac{s+y}{b}\right).$$

Thus (1. 7) becomes

$$\begin{aligned} (4. 1) \quad & \sum_{r(\bmod a)} \Phi\left(b\frac{r+x}{a} + y\right) \Phi\left(-\frac{r+x}{a}\right) + \sum_{s(\bmod b)} \Phi\left(-\frac{s+y}{b}\right) \Phi\left(x + a\frac{s+y}{b}\right) \\ &= \delta - \Phi(-x)\Phi(-y) - \frac{a}{2b} \Psi_2(y) - \frac{b}{2a} \Psi_2(x) - \frac{1}{2ab} \Psi_2(bx + ay). \end{aligned}$$

To show that (4. 1) is equivalent to (1. 4), we remark first that in the present case ($c=1$, $z=0$), $\delta=1$ if and only if $x=y=0$. If $x=y=0$, then since

$$\Phi(-x) = -\Phi(x) \quad (x \neq \text{integer}),$$

(4. 1) reduces to

$$\sum_{r=1}^{a-1} \Phi\left(\frac{br}{a}\right) \Phi\left(\frac{r}{a}\right) + \sum_{s=1}^{b-1} \Phi\left(\frac{s}{b}\right) \Phi\left(\frac{as}{b}\right) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right),$$

which is correct.

If $x=0$, $y \neq 0$, (4. 1) becomes

$$\begin{aligned} (4. 2) \quad \sum_{r=1}^{a-1} \Phi\left(\frac{br}{a} + y\right) \Phi\left(\frac{r+x}{a} + \sum_{s=0}^{b-1} \Phi\left(\frac{s+y}{b}\right) \Phi\left(a \frac{s+y}{b}\right) \right) = \\ = \frac{a}{2b} \Psi_2(y) + \frac{1}{2ab} \Psi_2(xay) + \frac{b}{2a} \Psi_2(0). \end{aligned}$$

If for some integer r_0 ,

$$(4. 3) \quad \frac{br_0}{a} + y = s_0,$$

where s_0 is an integer, it follows that

$$\frac{a(y-s_0)}{b} = -r_0, \quad \frac{r_0}{a} = -\frac{y-s_0}{b}.$$

Thus (4. 2) is in agreement with (1. 4). If (4. 3) is not satisfied there is of course no difficulty. The case $x \neq 0$, $y=0$ is handled in exactly the same way.

Finally let $xy \neq 0$. Then if for some integer r_0 , we have

$$(4. 4) \quad b \frac{r_0+x}{a} + y = s_0,$$

where s_0 is an integer, it follows that

$$x + a \frac{y-s_0}{b} = -r_0, \quad \frac{r_0+x}{a} = -\frac{y-s_0}{b}.$$

Thus (4. 1) agrees with (1. 4). If (4. 4) is not satisfied there is no difficulty.

Therefore, in all cases, (4. 1) agrees with (1. 4).

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References

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