Varieties of groups satisfying one two-variable law

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1. Introduction

If each pair x, y of elements of a group G satisfies the law $xyx^2y^2=1$, then, in particular, for $y=1$ the law reduces to $x^3=1$ so that G has exponent 3. Thus, the law is equivalent to the law $xyx^{-1}y^{-1}=1$ and G is abelian. More generally, we will say that G belongs to the variety $T_{a,b}$ if each pair of elements x,y of G satisfies the law

$$x^a y^a x^{a+1} y^{a+1} x^{a+2} \dots x^{a+b} y^{a+b} = 1, \tag{1.1}$$

for integers a and b. The class of such varietes is rather wide including, for example, all groups with finite exponent (Theorem 5.1).

In this paper*) we will investigate properties of groups belonging to $T_{a,b}$ for various values of a and b. The special cases $T_{a,2}$ and $T_{a,3}$ are considered in Section 2. In Section 3, upper bounds for the engel lengths of the soluble p-groups in $T_{1,p-2}$, p prime, are found. In Section 4 we investigate splitting properties of the finite groups of $T_{1,k}$ for various k. Finally, in Section 6 laws of the form (1.1) but with arbitrary constants are considered. These groups, however, do not form a variety in general, since subgroups need not satisfy such laws.

2. Short laws

In this section we investigate varietes of groups associated with laws of the type (1.1) with six and eight terms. The notation $[x, y, z, \dots]$ will be used for the left-normed commutator $[[x, y], z, \dots]$, where $[x, y]=x^{-1}y^{-1}xy$.

Lemma 2.1. *Let G be a group. The following statements on G are equivalent:*

(I) $x^a y^b x^c y^d x^e y^f = 1$ *for all* $x, y \in G$, *where* $a+c+e=b+d+f=0$;
(II) $[x^a, y^b]=[x^{-e}, y^{-d}]$ *for all* $x, y \in G$;
(III) $[y^b, x^c]=[y^{-f}, x^{-e}]$ *for all* $x, y \in G$;
(IV) $[x^c, y^d]=[x^{-a}, y^{-f}]$ *for all* $x, y \in G$.

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PROOF. We first show the equivalence of (I) and (II). From (I) it follows that

$$x^a y^b = y^{-f} x^{-e} y^{-d} x^{-c},$$

$$y^{-b} x^a y^b x^{-a} = y^{-f-b} x^{-e} y^{-d} x^{-c-a} = y^d x^{-e} y^{-d} x^e,$$

or,

$$[y^b, x^{-a}] = [y^{-d}, x^e],$$

and substitution of x^{-1} for x yields

$$[x^a, y^b] = [x^{-e}, y^{-d}].$$

Since the above operations are reversible, (I) and (II) are equivalent. The equivalence of (III) and of (IV) with (I) follows immediately from the remark that $x^a y^b x^c y^d x^e y^f = 1$ if and only if $y^b x^c y^d x^e y^f x^a = x^c y^d x^e y^f x^a y^b = 1$.

Lemma 2. 2. *Let G be a group and a, b, c, d, e, f, six integers such that $a+c+e = b+d+f=0$ and $(a, d)=(b, e)=(c, f)=(a, c)=(b, d)=1$. If for any pair of elements $x, y \in G$, $x^a y^b x^c y^d x^e y^f = 1$, then the elements whose orders divide some power of any one of the exponents form an abelian normal subgroup of G.*

PROOF. Since the identity is symmetric in each of the exponents, it is sufficient to show that two elements $u, v \in G$ commute whenever their orders divide some power of a. Thus, let the order of u be a divisor of a^n and the order of v be a divisor of some arbitrary power of a. We proceed by induction on n. The Lemma holds trivially if $n=0$. Assume the Lemma true for all u whose orders divide a^{n-1} and let w be an element of order dividing a^n. Then w^a commutes with all elements of orders dividing any power of a and , in particular, with v. Applying (II) and (IV) of Lemma 2. 1 to w and v, we obtain

$$1 = [w^{-e}, v^{-d}] = [w^{-c}, v^{-d}].$$

By hypothesis, $(a, d)=(a, c)=1$ so that the order of v is relatively prime to d and the order of w is relatively prime to c. Hence, $[w, v]=1$, as required.

For the following theorem the notation $x^y = y^{-1} x y$ will be used.

Theorem 2. 3. *If all elements x, y of a group G satisfy the law $x^a y^b x^c y^d x^e y^f = 1$, where $a+c+e=b+d+f=0$ and $(a, d)=(b, e)=(c, f)=(a, c)=(b, d)=1$, then G satisfies the second engel condition and G' is of exponent dividing $ab-de$.*

PROOF. By (II) of Lemma 2. 1 we have

$$[x^a, y^b] = [x^{a+c}, y^{-d}].$$

Since $[x^{a+c}, y^{-d}]=[y^{-d}, x^{-a-c}]^{x^{a+c}}$, it follows that $[x^a, y^b]=[y^b, x^{-a}]^{x^{a+c}}$ so that $[x^a, y^b]=[x^a, y^b]^{x^c}$ or

$$[x^a, y^b, x^c] = 1. \tag{2. 1}$$

Next, the law $[x^\alpha, y^\beta, x^\gamma]=1$ for integers α, β, γ, implies

$$[x^{-\alpha}(x^\alpha)^{y^\beta}, x^\gamma] = 1 = [(x^\alpha)^{y^\beta}, x^\gamma]$$

and $[x^\alpha, (x^\gamma)^{y^{-\beta}}]=1$ or $[(x^\gamma)^{y^{-\beta}}, x^\alpha]=1$, so that

$$[x^{-\gamma}(x^\gamma)^{y^{-\beta}}, x^\alpha] = 1 = [x^\gamma, y^{-\beta}, x^\alpha]. \tag{2. 2}$$

Hence, (2. 1) implies that

$$[x^c, y^b, x^a] = 1. \tag{2.3}$$

Exchanging c, b, a by a, f, e we obtain

$$[x^a, y^f, x^e] = 1. \tag{2.4}$$

It follows from (2. 1) and (2. 4) that $[x^a, y^{bf}, x^c] = [x^a, y^{bf}, x^e] = 1$, whence, since $(c, e) = (c, a) = 1$,

$$[x^a, y^{fb}, x] = 1$$

and by the above remark this implies that

$$[x, y^{bf}, x^a] = 1. \tag{2.5}$$

Replacing b, f, a by f, d, e we obtain

$$[x, y^{df}, x^e] = 1, \tag{2.6}$$

whence, (2. 5) and $(a, e) = (a, c) = 1$ imply that

$$[x, y^{bdf}, x] = 1. \tag{2.7}$$

It follows from Kappe's equation (6a) in [2], that if u is an element such that $[x, u, x] = 1$ for all x in G, then $[w, u^2, v, x] =$ for all $x, v, w \in G$. As bdf is divisible by 2, we have that $G^{b^2d^2f^2} \subseteq Z_3(G)$.

We next consider $G/G^{b^2d^2f^2}$. This quotient group is the product of its normal subgroups generated by the elements dividing b^2, d^2 and f^2, respectively. By Lemma 2. 2 these normal subgroups are abelian, and as $(b, d) = (d, f) = (f, b) = 1$, the intersection of any two of them is trivial. Hence, $G/G^{b^2d^2f^2}$ is abelian so that

$$G' \leqq G^{b^2d^2f^2} \leqq Z_3(G).$$

This shows that G is nilpotent of class 4 so any commutator of weight 5 is equal to 1. By (2. 5) we obtain in particular that

$$1 = [x, y^{bf}, x^a, x] = [x, y, x, x]^{baf} \quad \text{and}$$

$$1 = [x, y^{bf}, x^a, y] = [x, y, x, y]^{baf}.$$

By the cyclical arrangement of the integers a, b, c, d, e, f, we have further $1 = [x, y, x, x]^{afe} = [x, y, x, x]^{fed} = [x, y, x, x]^{edc}$, and by the divisibility conditions on a, b, c, d, e, f, we find that baf, afe, fed, edc have no common divisor. Hence,

$$[x, y, x, x] = [x, y, x, y] = 1. \tag{2.8}$$

Using (2. 5) again and applying the cyclicity of the integers we find

$$1 = [x, y^{bf}, x^a] = [x, y, x]^{fab} = [x, y, x]^{afe} = [x, y, x]^{fed} = [x, y, x]^{edc},$$

which in turn yields

$$[x, y, x] = 1. \tag{2.9}$$

Finally, applying (II) of Lemma 2. 1, we derive

$$1 = [x^a, y^b][x^{-e}, y^{-d}] = [x, y]^{ab-de}$$

so that elements of the abelian group G' have an order dividing $ab-de=ef-be=cd-af$, where this equality is a consequence of the hypothesis $a+c+e=b+d+f=0$.

It is clear that the converse of this theorem is also true.

Corollary 2. 4. Let G be a group each pair x, y of whose elements satisfies the law

$$x^a y^a x^{a+b} y^{a+b} x^{a+2b} y^{a+2b} = 1;\ a, b \text{ integers},\ (a, b) = 1.$$

Then G' has exponent dividing $(3, b)$.

Proof. From the Theorem it follows that if a group G satisfies the law $x^a y^a x^{a+b} y^{a+b} x^{-2a-b} y^{-2a-b} = 1$, $(a, b) = 1$, then G' has exponent dividing $(a+b)^2 - a(-2a-b)$. The group G of the Corollary staisfies the latter law since it has exponent $a+(a+b)+(a+2b)=3a+3b$, whence $x^{a+2b}=x^{-2a-b}$ for any $x \in G$. Thus, G' has exponent dividing $(a+b)^2 - a(a+2b) = b^2$, and since $(a, b)=1$ it follows that G' has exponent dividing $(3a+3b, b^2)=(3, b)$.

Theorem 2. 5. *Let G be a group satisfying the law*

$$x^{a+1} y^{a+1} x^{a+2} y^{a+2} x^{a+3} y^{a+3} x^{a+4} y^{a+4} = 1, \quad a \text{ an integer.} \tag{2. 10}$$

Then G is nilpotent of class at most 2. In particular, if $(5, a)=1$, $a \neq 0$, then G is abelian.

Proof. Choosing $y=1$ in (2. 10), it follows that $x^{4a+10}=1$ so that G has exponent $2(2a+5)$. Let $x^{2a+5}=y^{2a+5}=1$. The law (2. 10) applied to x^2 and y^2 then yields that

$$x^{-3} y^{-3} x^{-1} y^{-1} xyx^3 y^3 = 1, \quad \text{or} \quad [x, y][x^{-3}, y^{-3}] = 1. \tag{2. 11}$$

If (2. 11) is applied to the identity $[x^{-3}, y^{-3}]=[y^{-3}, x^3]^{x^{-3}}$, we obtain $[y, x]=[x^{-1}, y]^{x^{-3}}=[y, x]^{x^{-4}}$ so that $[x, y, x^4]=1$, and since $(4, 2a+5)=1$ this implies further that $[x, y, x]=1$. Thus, $[x^m, y^n]=[x, y]^{mn}$ for integers m, n so that $(xy)^{2a+5}=x^{2a+5}y^{2a+5}$ and xy has order dividing $2a+5$ also. Hence, the elements with orders dividing $2a+5$ form a normal subgroup H of G satisfying (2. 11). Let $2a+5=3^k \cdot b$, where $(3, b)=1$. By (2. 11), the elements of orders dividing 3^k form a normal abelian subgroup H_1 of H. From Levi [6], it follows from $[x, y, x]=1$ that the elements whose orders divide b form a normal subgroup H_2 of H which is nilpotent of class at most 2. Thus H, as the direct product of H_1 and H_2, is nilpotent of class at most 2. In particular, (2. 11) for elements $x, y \in H$ can be rewritten in the form $[x, y]^{10}=1$, so that if $(b, 5)=1$, H_2, and hence H, if abelian.

Finally, let $x^2=y^2=1$. The law (2. 10) applied to x, y shows that $(xy)^2=1$. so that the elements of G of order 2 form a normal subgroup K of G, and K is abelian Clearly, G is the direct product of H and K, so that G is of the same nilpotency class as H.

3. The variety $T_{1,p-2}$

In this section we investigate the soluble p-groups of the variety $T_{1,p-2}$ for an odd prime p. The special cases $p=3$ and $p=5$ have been discussed in the Introduction and Section 2, respectively.

Theorem 3. 1. *Let G be a metabelian p-group satisfying the law*

$$xyx^2y^2 \dots x^{p-1}y^{p-1} = 1, \quad \text{for every} \quad x, y \in G. \tag{3. 1}$$

Then G is nilpotent of class at most $(p+1)/2$ and every two generator subgroup of G is nilpotent of class at most $(p-1)/2$.

Conversely, if G is metabelian of exponent p and if every two generator subgroup is nilpotent of class at most $(p-1)/2$, then G satisfies the law (3. 1), *i. e., G belongs to $T_{1,p-2}$.*

PROOF. The proof of the first part of the theorem is divided into two parts.

Part I: Applying (3. 1) to the pair $[x, y], z^{-1}$ for any $x, y \in G$ yields

$$[x, y]\, z^{-1}\, [x, y]^2 z^{-2} \dots [x, y]^{p-1} z^{-p+1} = 1,$$

or

$$\prod_{n=1}^{p-1} z^{-g(n)} [x, y]^n z^{g(n)} = 1, \tag{3. 2}$$

where $g(n) = n(n-1)/2$. In terms of the ring $E(G, G')$ of endomorphisms induced in G' by the inner automorphisms of G, (3. 2) can be written in the more concise form

$$F(z) = \sum_{n=1}^{p-1} n z^{g(n)} = 0. \tag{3. 3}$$

By hypothesis, $E(G, G')$ has characteristic p, and for any $z \in G$, $z^p = 1$. Set $s = (p-1)/2$. Then the term in (3. 3) with coefficient $n = (s+1) - k$ is associated with the exponent

$$g(s+1-k) = \frac{1}{8}\left(p^2 - 4kp + (2k-1)(2k+1)\right)$$

and can be added to the term with coefficient $n = (s+1) + k$ which is associated with the exponent

$$g(s+1+k) = \frac{1}{8}\left(p^2 + 4kp + (2k-1)(2k+1)\right) \equiv g(s+1-k) \pmod{p}.$$

Part I consists in showing that G satisfies the $(p+1)/2 = (s+1)$ engel condition. For this it is sufficient to show that $F(z)$ is not divisible by $(1-z)^{s+1}$ since $(1-z)^p = 1 - z^p = 0$ in $E(G, G')$. In fact, it will be shown that $F(z)$ is divisible by $(1-z)^s$.

In general, for $H(z) = \sum a_n z^n$, the first two derivatives of H are

$$H'(z) = \sum a_n n z^{n-1}$$

$$H''(z) = \sum a_n n(n-1) z^{n-2},$$

so that $H'(1) + H''(1) = \sum a_n n^2$. It follows by an easy induction that for any $k \geqq 1$ there exists a linear combination of $H'(1), H''(1), \dots, H^{(k)}(1)$ equal to $\sum a_n n^k$. Thus, there is a linear combination of the first k derivatives of $F(z)$ which for $z = 1$ is

$$\sum_{n=1}^{s} (g(n))^k + (s+1)(g(s+1))^k. \tag{3. 5}$$

The objective of Part I will be achieved by showing that (3. 5) is divisible by p for $1 \leqq k < s$ and is not divisible by p for $k = s$. (One sees easily that $F(1) = 0$.)

Since p is odd there is no loss in generality by considering

$$F_k = 2 \cdot 8^k \sum_{n=1}^{s} (g(n))^k + (-1)^k \tag{3.6}$$

instead of (3. 5). Further expansion shows that

$$F_k = 2 \sum_{n=1}^{s} (4n^2 - 4n)^k + (-1)^k = 2 \sum_{n=1}^{s} ((2n-1)^2 - 1)^k + (-1)^k,$$

and since $[2n-1]^2 \equiv [2(p-(n-1))-1]^2 \pmod{p}$, we have

$$F_k \equiv \sum_{n=1}^{p-1} [(2n-1)^2 - 1]^k - (-1)^k + (-1)^k = \sum_{n=1}^{p-1} (4n^2 - 4n)^k.$$

By considering the identity $\sum_{n-1}^{p-1} \binom{n}{k} = \binom{p}{k+1}$ for $k = 0, 1, \ldots p-1$, it follows easily that $\sum_{n=1}^{p-1} n^k \equiv 0 \pmod{p}$ for $k < 2s = p-1$ but $\equiv -1 \pmod{p}$ for $k = 2s$. Thus, for $k < s$, $\sum_{n=1}^{p-1} (4n^2 - 4n)^k \equiv 0$, while for $k = s$, $\sum_{n=1}^{[p-1} (4n^2 - 4n)^k \equiv -(4)^{p-1} \equiv -1$, which completes the proof of Part I.

Part II: The elements $t \in E(G, G')$ for which $[x, y]^t = 1$, $t \in G$, for a fixed pair $x, y \in G$, form an ideal E_{xy} in $E(G, G')$. From Part I it follows that $(1-z)^s \in E_{xy}$ for every $z \in G$. The proof of the first part of the Theorem will be completed by showing that for any non-negative integers a, b $(1-x)^a(1-y)^b \in E_{xy}$ if $a+b = s-1$.

We now use the identity (3. 1) for x^{-1} and y^{-1}. In the center of the identity we find the commutator $x^{-s}y^{-s}x^s y^s$. Next, we consider the central eight powers

$$x^{-s-1}y^{-s-1}x^{-s}y^{-s}x^s y^s x^{s+1} y^{s+1} = [x^{s+1}, y^{s+1}]\, y^{-s-1} x^{-s-1} [x^s, y^s] x^{s+1} y^{s+1}.$$

Further applications of this method, next considering the central 16, 24, etc., powers, leads finally to writing the left side of the identity in the form of a product of commutators and their conjugates. More precisely, we obtain

$$1 = x^{-1}y^{-1}x^{-2}y^{-2} \ldots x^2 y^2 xy = \prod_{n=1} y^{-g(n)} x^{-g(n)} [x^n, y^n] x^{g(n)} y^{g(n)}, \tag{3.7}$$

as can be shown by induction using the fact that G is metabelian so that, for any integers a, b, c, d, k,

$$y^{-a}x^{-b}(y^{-c}x^{-d}[x^k, y^k]x^d y^c)x^b y^a = y^{-a-c}x^{-b-d}[x^k, y^k]x^{b+d}y^{a+c}.$$

Thus, since $[x^n, y] = [x, y][x, y]^x [x, y]^{x^2} \ldots [x, y]^{x^{n-1}}$ in a metabelian group, it follows from (3. 7) that

$$f(x, y) = \sum_{n=1}^{s} x^{g(n)} y^{g(n)} (1 + x + \ldots + x^{n-1})(1 + y + \ldots + y^{n-1}) \in E_{xy}. \tag{3.8}$$

In particular, $f(x, 1)=\sum_{n=1}^{s} nx^{g(n)}(1+x+\ldots+x^{n-1})$, so that

$$(1-x)f(x,1) = \sum_{n=1}^{s} nx^{g(n)}(1-x^n) = \sum_{n=1}^{s} \{nx^{g(n)} - nx^{g(n)-n}\} =$$

$$= \sum_{n=1}^{s} x^{g(n)} - sx^{g(s)+1} = F(x). \tag{3.9}$$

It follows from Part I that $f(x, 1)$ has a factor $(1-x)^{s-1}$ but not $(1-x)^s$. Thus, after substituting for x^s in $f(x, 1)$ using $(1-x)^s=0$, the degree of $f(x, 1)$ will not be less than $s-1$. Hence, $f(x, y)$ with $(1-x)^s=(1-y)^s=0$ reduces to a polynomial

$$f(x,y) = \sum_{n=1}^{s-1} f_n(y)x^n, \quad \text{where} \quad f_{s-1}(1) \neq 0. \tag{3.10}$$

E_{xy} has the further property that if $zt \in E_{xy}$ for any $z \in G$, then $t \in E_{xy}$. In addition, for any polynomial $h(x, y) \in E_{x,y}$, we have that $h(yx, y) \in E_{xy}$ since $[yx, y]=[x, y]$. It follows that if

$$h(x,y) = \sum_{i=0}^{m} h_i(y)x^i \in E_{xy} \tag{3.11}$$

then

$$h(yx,y) = \sum_{i=0}^{m} h_i(y)y^i x^i \in E_{xy},$$

and

$$h(x,y) - h(yx,y) = \sum_{i=0}^{m} h_{i,1}(y)(1-y^i)x^i \in E_{xy},$$

so that

$$x^{-1}[h(x,y) - h(yx,y)] = \sum_{i=1}^{m} h_{i,1}(y)x^{i-1} \in E_{xy}. \tag{3.12}$$

Moreover, if $m<p$ and $h_m(y)=(1-y)^j h'_m(y)$, where $h'_m(1)\neq 0$, then

$$h_{m,1}(y) = (1-y)^{j+1}(1+y+\ldots+y^{m-1})h'_m(y) = (1-y)^{j+1}h'_{m,1}(y),$$

where $h'_m(1)\neq 0$.

The above remarks may now be applied to $f(x, y)$ as given by (3. 10). Repeating the above sequence of steps from (3. 11) to (3. 12) $s-1$ times yields finally a polynomial $f_{s-1,s-1}(y)x^0$ in y only which will be divisible by $(1-y)^{s-1}$ but not by $(1-y)^s$. Since $(1-y)^s \in E_{xy}$, this implies that $(1-y)^{s-1} \in E_{xy}$. Hence, G satisfies an s engel condition.

Now a metabelian group of exponent p which satisfies an m engel condition for any $m<p$ has the property that any two generator subgroup is nilpotent of class at most m. To see this, set $x=1+X$, $y=1+Y$. Then, $X^{m-1}=Y^{m-1}=0$ (mod E_{xy}). The substitution of yx for x corresponds to the substitution of $X+xY$ for X. Thus, $(X+xY)^{m-1} \in E_{xy}$ so that $\sum_{k=1}^{m-1} \binom{m}{k} X^k Y^{m-1} x^{m-k} \in E_{xy}$. Continuing this process, substituting $X+xY$ for X and using the fact that the polynomial before

the substitution was in E_{xy}, one arrives finally at the conclusion that XY^{m-2}, $X^{m-2}Y \in E_{xy}$, and by comparing with the various polynomials in the steps which led to this result one obtains the end conclusion that $X^aY^b = X^aY^b = 0 \pmod{E_{xy}}$ for any nonnegative integers a, b for which $a+b=m-1$. It follows from these remarks that every two generator subgroup of G is nilpotent of class at most s.

To show that G is nilpotent of class at most $s+1$, we consider $Z^s=(1-z)^s=0$ in $E(G, G')$ for any $z \in G$. The substitution wz for z, $w \in G$, corresponds to a substitution $Z+zW$ for Z where $W=1-w$. As before, by considering first $Z^s=W^s=1$, then $(Z+zW)^s=0$ etc., one sees that $Z^aW^b=0$ for any nonnegative integers a, b such that $a+b=s$. The proof now proceeds by induction, next substituting $z_1z_2z_3$ for z, then observing that $(1-z_1)^a(1-z_2)^b(1-z_3)^c=0$ for $a+b+c=s$, then substituting $z_1z_2z_3z_4$ for z, etc., and one finally obtains that $(1-z_1)^a \ldots (1-z_k)^m=0$ if $a+\ldots \ldots+m=s$. Thus, G is nilpotent of class at most $s+1$ (see [7]).

For the converse we note that (3. 1) is equivalent to $f(x, y) \in E_{xy}$. With $X=x-1$, $Y=y-1$, $f(x, y)$ can be written in the form

$$f(1+X, 1+Y) = \sum_{n=1}^{s} (1+X)^{g(n)} (1+Y)^{g(n)} X^{-1}\left((1+X)^n-1\right) Y^{-1}\left((1+Y)^n-1\right)$$

or

$$f(x, y) = \sum_{n=1}^{s} X^{-1}\{(1+X)^{g(n+1)}-(1+X)^{g(n)}\} Y^{-1}\{(1+Y)^{g(n+1)}-(1+Y)^{g(n)}\}.$$

The coefficient of $X^{k+1}Y^{j-1}$ in this expression is

$$\sum_{n=1}^{s} \left\{\binom{g(n+1)}{k}-\binom{g(n)}{k}\right\}\left\{\binom{g(n+1)}{j}-\binom{g(n)}{j}\right\}.$$

By hypothesis, $X^aY^b=0$ if $a+b=(p-3)/2$. Hence, we may assume that $k=1+j-1 \leqq (p-5)/2$, that is, $k+j \leqq s$.

As a polynomial in n, $\binom{g(n+1)}{k}-\binom{g(n)}{k}$ is an odd function of n. To see this, it suffices to consider the expression

$$\varphi(n) = \left(\frac{n(n+1)}{2}\right)^m - \left(\frac{n(n-1)}{2}\right)^m$$

for arbitrary $m>0$. In particular,

$$\varphi(n) = \frac{n^m}{2^m}\{(n+1)^m-(n-1)^m\} = \frac{n^m}{2^m}\left\{2 \sum_{\substack{i=1 \\ i,\ \text{odd}}}^{m}{}' \binom{m}{i} n^{m-i}\right\} = \frac{1}{2^{m-1}} \sum_{\substack{i=1 \\ i,\ \text{odd}}}^{m}{}' \binom{m}{i} n^{2m-i}$$

as desired. Thus, the coefficient of $X^{k-1}Y^{j-1}$ involves even powers of n only, and it follows from previous remarks that this coefficient is divisible by p since $k+j \leqq s$. Hence, $f(x, y)$ is divisible by p so that $f(x, y)=0$ in $E(G, G')$ so that G satisfies the law (3. 1). This completes the proof of the Theorem.

Corollary 3. 2. Let G be a metabelian group of exponent p. If every two generator subgroup of G is nilpotent of class at most $(p-1)/2$, then G is nilpotent of class at most $(p+1)/2$ (cp. [7]).

Corollary 3. 3. Let G be a p-group satisfying (3. 1) and $x \in G$ be an element whose normal closure (i. e., the least normal subgroup containing x) N_x is abelian. Then x is a right engel element (cf., GRUENBERG [1]); in particular, for any $z \in G$,

$$[x, \underbrace{z, \ldots, z}_{s \text{ times}}] = 1, \quad \text{where} \quad s = \frac{p-1}{2}$$

PROOF. By hypothesis, $[x, y]^{w(x,y)} = [x, y]^{w(1,y)}$, where y is an arbitrary element of G and $w(x, y)$ any element of $gp(x, y)$. Further, for $u, v \in G$, $[x, z]^u$ commutes with $[x, y]^v$. It follows that the automorphisms induced on $N'_{x,z} = gp([x^a, z^b]) \leqq N_x$, where a and b run through all integers, by the inner automorphisms associated with x and z generate a commutative ring $E(x, z: N'_{x,z})$ of endomorphisms of $N'_{x,z}$. As in (3. 7), the law (3. 1) applied to x^{-1} and z^{-1} yields

$$1 = x^{-1}z^{-1}x^{-2}z^{-2} \ldots x^2z^2xz = \prod_{n=1}^{s} z^{-g(n)}[x^n, z^n]z^{g(n)} =$$

$$= \prod_{n=1}^{s} z^{-g(n)}[x, z^n]^n z^{g(n)},$$

which, in terms of $E(x, z; N'_{x,z})$ can be expressed in the form

$$f(z, 1) = \sum_{n=1}^{s} nz^{g(n)}(1 + z + \ldots + z^{n-1}) = 0.$$

As shown in the proof of Theorem 3. 1, $f(z, 1)$ has a factor of $(1-z)^{s-1}$ but not $(1-z)^s$. Since $1+z+\ldots+z^{p-1} = (1-z)^{p-1} = 0$ in $E(x, z; N'_{x,z})$ (that is, $[x, z^p] = 1$), it follows that $(1-z)^{s-1} = 0$ in $E(x, z; N'_{x,z})$, which proves the Corollary.

Corollary 3. 4. Let G be a soluble p-group of soluble length n which satisfies (3. 1). Then G satisfies an engel condition of length at most $(n-1)s$, where $s = (p-1)/2$.

PROOF. (Induction on n). The case $n = 2$ has been proved correct in Theorem 3. 1. Thus, let G be a soluble group of derived length $k > 2$ and assume the Corollary true for groups of derived length $\leqq k-1$. By hypothesis, $G/G^{(k-1)}$ satisfies an engel condition of length at most $(n-2)s$. Hence, any commutator $[x, \underbrace{y, \ldots, y}_{(n-2)s \text{ times}}] \in$ $\in G^{(k-1)}$, for arbitrary $x, y \in G$. Further, $G^{(k-1)}$ is abelian so that by Corollary 3. 3, $[z, \underbrace{y, \ldots, y}_{s \text{ times}}] = 1$ for any $z \in G^{(k-1)}$. Thus, for any $[x, \underbrace{y, \ldots, y}_{(n-1)s \text{ times}}] = 1$, which proves the Corollary.

4. Spitting properties

In this section we investigate splitting properties which are consequences of the identities studied here. We begin with a transparent situation.

Lemma 4. 1. *Let G be a finite group whose elements satisfy the identity $xy \ldots x^n y^n = 1$, and let N be an abelian normal subgroup of G of exponent s such that G/N is cyclic of order t and $(sn, t(n+1)) = 1$ or $(tn, s(n+1)) = 1$. Then G is abelian.*

PROOF. Let gN be a generator of G/N and h an arbitrarily chosen element of N. The identity applied to g^{-1} and gh yields

$$g^{-1}(gh)g^{-2}(gh)^2 \ldots g^{-n}(gh)^n = 1. \tag{4.1}$$

In terms of the endomorphism ring induced by G in N, (4.1) can be expressed in the form

$$1+(g+1)+\ldots+(g^{n-1}+\ldots+g+1)=0. \tag{4.2}$$

Multiplying (4.2) by $g-1$ gives

$$0=(g-1)+(g^2-1)+\ldots+(g^n-1)=(g^n+g^{n-1}+\ldots+g+1)-(n+1),$$

which after a further multiplication by $g-1$ becomes

$$0 = (g^{n+1}-1)-(n+1)(g-1). \tag{4.3}$$

Taking $y=1$ in the original identity, we obtain that G has exponent dividing $n(n+1)/2$. The two number-theoretic conditions indicated in the statement of the Lemma lead to two cases.

Case 1: s divides n and t divides $n+1$.

Case 2: s divides $n+1$ and t divides n.

For Case 1, (4.3) simplifies to

$$0 = (g^{n+\frac{1}{a}}-1)-(n+1)(g-1) = 0-(g-1),$$

that is, g centralizes N. For Case 2 we obtain correspondingly

$$0 = (g^{n+1}-1)-(n+1)(g-1) = (g-1)-0,$$

and again g centralizes N. Thus, in either case G is abelian.

Corollary 4.2. If each pair x, y of elements of the finite group G satisfy the equation $xyx^2y^2 \ldots x^ny^n=1$, and if G is a pq-group, where p is a prime dividing n and q is a prime dividing $n+1$, then G is nilpotent.

PROOF. Assume that G is a minimal counterexample to the Corollary. By a result of IWASAWA, SCHMIDT, RÉDEI (see Rédei [3], *p.* 304) such a group is of the form given in Lemma 4.1. It follows that G is abelian and therefore nilpotent, contrary to assumption. This proves the Corollary.

Theorem 4.3. *If G is finite soluble group satisfying the law $xy \ldots x^ny^n = 1$, then G is the direct product of G^n and G^{n+1}.*

PROOF. We proceed by induction on the order of G. As the exponent of G is a divisor of $n(n+1)/2$, the Theorem is true if G is abelian. Assume now that H is a minimal counterexample. If N is a minimal normal subgroup of H the Theorem is true for H/N. Thus, $H/N = A/N \otimes B/N$, where $A/N=(H/N)^n$, $B/N=(H/N)^{n+1}$. If the exponent of N is a divisor of n, then the orders of N and A/N are relatively prime. By Schur's Theorem (see, for example, Zassenhaus [5], Theorem 25, p. 162) there is a subgroup A_1 of A such that $A_1N=A$ and $A_1 \cap N=1$. By Lemma 4.1, any element of A_1 centralizes N, and we obtain $A=A_1 \otimes N$ and $G=A_1 \otimes B$. The reader will check without difficulty that $A_1=G^n$ and $B=G^{n+1}$. If the exponent of N is a divisor of $n+1$, we proceed similarly.

Theorem 4. 4. *Let G be a finite group each pair x, y of whose elements satisfy the identity $xy\ldots x^n y^n = 1$, where n or $n+1$ is a power of an odd prime p. Then G is the direct product of its p-Sylow subgroup and its complement.*

PROOF. We prove first by induction on the order of G that G has a normal p-complement. Thus, assume that G has no p-complement and G is the smallest such group. Then all proper subgroups and quotient groups of G possess a normal p-complement. If G contains a normal p-subgroup, then this normal p-subgroup N is the p-Sylow group of G, and by the theorem of Schur (Zassenhaus [5]) there is a complement C of N in G. By Lemma 4. 1, the elements of $C\varphi(N)/\varphi(N)$ permute with $N/\varphi(N)$, where $\varphi(N)$ is the Frattini subgroup of N. But the elements of C have orders prime to p so that the elements of C permute with those of N and G is the direct product $N \otimes C$. Therefore, our minimal counterexample G does not contain a normal p-subgroup different from 1. Hence, Thompson's Theorem (see, for instance, SCHENKMAN [4], p. 273) is applicable, since the normalizers and centralizers of certain subgroups have a normal p-complement by the minimality of G. But then G has a normal p-complement, contrary to our hypothesis. Consequently, all groups which satisfy the given identy have a normal p-complement. Denote this normal p-complement by C, and denote a fixed p-Sylow subgroup by S. If T is a Sylow subgroup of C, the normalizer $N(T)$ of T contains a p-Sylow subgroup of G since $CN(T) = G$. The normalizer of a conjugate $\overline{T}$ of T will therefore contain S, and by Lemma 4. 1, S will centralize $\overline{T}$. Consequently, for each prime q dividing the order of C there is a q-Sylow subgroup which is centralized by S. This implies immediately that S and C commute so that $G = S \otimes C$, as desired.

5. Examples

Every finite group satisfies an identity of the form studied in this paper. More generally, we have the following result.

Theorem 5. 1. *If G is a group of finite exponent k, then every pair x, y of elements of G satisfies the identity*

$$xyx^2y^2\ldots x^{k^2}y^{k^2} = 1. \tag{5. 1}$$

PROOF. By hypothesis, the elements x and y have orders dividing k. It follows that

$$xyx^2y^2\ldots x^{k^2}y^{k^2} = (xyx^2y^2\ldots x^{k-1}y^{k-1}x^ky^k)^k = 1,$$

which proves the Theorem.

Remark. There is no bound to the nilpotency class of groups satisfying the identity (5. 1) for $k = p$, where p is an odd prime, since this identity, by Theorem 5. 1, is satisfied in all extensions of abelian p-groups of exponent p by abelian p-groups of exponent p, the nilpotency class of which is not bounded (see, for instance, Gruenberg [1], p. 166).

Theorem 5. 2. *If G is metabelian and possesses an abelian normal subgroup N of exponent m such that G/N is abelian and of exponent k, then every pair x, y of elements of G satisfies the identity*

$$xyx^2y^2 \ldots x^{4mk}y^{4mk} = 1. \tag{5. 2}$$

PROOF. We denote $xyx^2y^2 \ldots x^{2k}y^{2k}$ by a. Since G/N is abelian and of exponent k we observe that $a \equiv 1 \bmod N$ and $a \in N$. Further, x^k, y^k and all their conjugates are elements of the abelian normal subgroup N. Thus, by the commutativity of N and G/N, we have

$$x^{2k+1}y^{2k+1} \ldots x^{4k}y^{4k} = (xy \ldots x^{2k}y^{2k})b = ab,$$

where

$$b = (\Pi y^{\frac{i(i+1)}{2}} x^{2k} y^{-\frac{(i+1)}{2}})(\Pi x^{\frac{i(i+1)}{2}} y^{2k} x^{-\frac{i(i+2)}{2}}).$$

As both a and b are products of k-powers, they are both in N. By induction, we obtain

$$x^{2tk+k}y^{2tk+1} \ldots x^{2(t+1)k}y^{2(t+1)k} = x^{2(t-1)k+1}y^{2(t-1)k+1} \ldots x^{2tk}y^{2tk}b = a \cdot b^t,$$

so that

$$xyx^2y^2 \ldots x^{4mk}y^{4mk} = a^{2m}b^{\frac{2m2(m-1)}{2}} = 1,$$

which was to be shown.

Remark. If k and m in Theorem 5. 2 are odd, then

$$xyx^2y^2 \ldots x^{mk}y^{mk} = 1,$$

while if one of k or m is odd, then $xyx^2y^2 \ldots x^{2mk}y^{2mk} = 1$. For, if k is odd, we may replace the element a in the above proof by $xyx^2y^2 \ldots x^k y^k_a \in N$, and if m is odd, $m(m-1)/2$ is divisible by m.

6. Identities with parameters

We say that G satisfies the law $w(x, y; x_1, \ldots, x_n) = 1$ parametrically if there exist fixed elements $g_1, \ldots, g_n \in G$ such that each pair x, y of elements of G satisfy the law $w(x, y; g_1, \ldots, g_n) = 1$. In general, the class of groups which satisfy a law parametrically do not form a variety since the property does not extend to subgroups in general.

As noted in the Introduction, a group satisfying the law $xyx^2y^2 = 1$ is abelian and of exponent 3. This result admits the following generalization.

Theorem 6. 1. *Let G be a group satisfying the law*

$$xaybxcxdyeyf = 1 \tag{6. 1}$$

parametrically, where a, b, c, d, e, f are certain fixed elements in G. Then G satisfies the law $xyx^2y^2 = 1$.

PROOF. With $x = y = 1$ (6. 1) reduces to $abcdef = 1$. Next, with $y = a^{-1}b^{-1}$, (6. 1) gives

$$x^2cxda^{-1}b^{-1}ea^{-1}b^{-1}g = 1. \tag{6. 2}$$

It follows from (6. 2) that x^2cx is constant in G for every choice of x so that with $x = 1$, $x^2cx = c$. Thus, $cx = x^{-2}c$ for every $x \in G$ and (6. 1) is equivalent to

$$xaybxx^{-2}cdyeyf = 1. \tag{6. 3}$$

Thus, for any choice of y, say $y = g \in G$, $xagbx^{-1}$ is constant in G, so that agb is in

the center of G for any $g \in G$. Replacing g by $a^{-1}gb^{-1}$, it follows that g is in the center of G, that is, G is abelian. Since $abcdef = 1$, it follows from (6. 1) that $xyx^2y^2 = 1$ for any $x, y \in G$.

The following result generalizes Corollary 2. 4 (with $a = b = 1$) analogously.

Theorem 6. 2. *Let G satisfy the law*

$$xaybx^2cy^2dx^3ey^3f = 1, \quad \text{for all} \quad x, y \in G, \tag{6. 4}$$

parametrically for fixed elements $a, b, c, d, e, f \in G$. Then G is abelian with exponent 6.

PROOF. It will be convenient to divide the proof into several parts.

1) If y has order 2, then (6. 4) reduces to $xaybx^2cdx^3eyf = 1$ or $yfxaybx^2cdx^3e = 1$. Hence, for any fixed choice of $x \in G$, $yfxay$ is constant for any choice of y with $y^2 = 1$, so that, in particular, $yfxay = fxa$. Substituting $x = f^{-1}ga^{-1}$ yields that $ygy = g$, that is, y is central.

2) If y has order 3, then (6. 4) reduces to $xaybx^2cy^{-1}dx^3ef = 1$ or $ybx^2cy^{-1}dx^3efxa = 1$. With $y = 1$ this yields $bx^2cdx^3efxa = 1$ so that $ybx^2cy^{-1} = bx^2c$ for any $x, y \in G$ with $y^3 = 1$. In particular, $ybcy^{-1} = bc$. Substitute $b^{-1}zb$ for x: $yz^2bcy^{-1} = z^2bc$, so that $yz^2y^{-1} = z^2$, that is, y commutes with any square.

3) If y has order 4, then y^2 is central by Part 1 so that, by (6. 4), $xaybx^2cdx^3eyf = 1$ or $yfxay = (bx^2cdx^3e)^{-1} = fxa$. In particular, for $x = f^{-1}a^{-1}$ this yields that $y^2 = 1$ so that, again by Part 1, y is central.

4) If $y^6 = 1$, then it follows immediately from Parts 1 and 2 that y commutes with any square in G.

5) For $y = 1$, (6. 4) reduces to $xabx^2cdx^3ef = 1$ so that, since $(ab) = (ef)^{-1}(cd)^{-1}$ (set $x = 1$),

$$xw^{-1}v^{-1}x^2vx^3w = 1, \quad \text{where} \quad v = cd, \ w = ef. \tag{6. 5}$$

With the substitution $x = vzv^{-1}$, (6. 5) becomes

$$1 = vzv^{-1}w^{-1}z^2vz^3v^{-1}w \quad \text{or} \quad v^{-1}wvzv^{-1}w^{-1}v \cdot v^{-1}z^2vz^3 = 1. \tag{6. 6}$$

Since this is valid for any $z \in G$, comparing (6. 5) with (6. 6), identifying x with z, yields that $vwv^{-1}xv^{-1}w^{-1}v = wxw^{-1}$, so that $w^{-1}v^{-1}wv$ is central in G.

Thus, mod $Z(G)$, the center of G, $[w, v] = 1$ so that in $G/Z(G)$, (6. 5) yields, first with the substitution $x = v$ then with $x = w$, that $v^6 = w^6 = 1$. Thus, by Part 4, it follows that for any $x \in G$, $[x^2, v] = [x^2, w] = 1 \bmod Z(G)$, or $[x^2, v], [x^2, w] \in Z(G)$.

6) With $x = w$, (6. 5) reduces to

$$v^{-1}w^2vw^4 = 1, \tag{6. 7}$$

and with $x = v$ to

$$vw^{-1}v^5w = 1. \tag{6. 8}$$

From (6. 7) it follows that $v^{-1}w^6v = w^{-12}$, and since $w^6 \in Z(G)$, by Part 5, this implies that $w^{18} = 1$, so that by Part 1, $w^9 \in Z(G)$. Since $w^6 \in Z(G)$, this yields that $w^3 \in Z(G)$.

7) From (6. 7) and (6. 8) it follows that

$$v^{-1}w^2vw^4 \cdot w^{-1}v^5wv = 1, \quad \text{or} \quad w^3vw^3v^5 = 1.$$

Since $w^3 \in Z(G)$, this means that $w^6 = v^{-6}$ so that $w^{18} = v^{-18} = 1$ and $v^9 \in Z(G)$, by Part 1. Since $v^6 \in Z(G)$, this implies that $v^3 \in Z(G)$.

8) Mod $Z(G)$, $[x^2, v] = [x^2, w] = 1$ so that, by (6. 5),

$$xw^{-1}v^{-1}x^2wx^3w = 1 = xw^{-1}v^{-1}vx^5w = xw^{-1}x^5w = x^5wxw^{-1} = x^6[x, w^{-1}].$$

Substituting x^2 for x yields that $x^{12}[x^2, w^{-1}] = x^{12} = 1$, that is, $x^{12} \in Z(G)$.

9) Writing (6. 4) for x and y^4 yields $xay^4bx^2cy^8dx^3ey^{12}f = 1$, so that, by Part 8, $xay^4bx^2cy^{20}dx^3ef = 1$, or, $y^4bx^2cy^{20}dx^3efxa = 1$. Hence, $y^4bx^2cy^{20} = bx^2c$. In particular, $y^4bcy^{20} = bc$ so that $cy^{20} = b^{-1}y^{-4}bc$. Therefore, it follows that $y^4bx^2cy^{20} = y^4bx^2b^{-1}y^{-4}bc = bx^2c$, or, $y^4bx^2b^{-1}y^{-4} = bx^2b^{-1}$. Substituting $b^{-1}zb$ for x, we have that $y^4z^2y^{-4} = x^2$, that is, $[z^2, y^4] = 1$ for any $z, y \in G$.

10) From Parts 5 and 9 it follows that for any $x \in G$, $[x^2, v] = [x^2, w] = 1$ Thus, (6. 5) is equivalent to $xw^{-1}x^5w = 1$. Hence, $x^5wxw^{-1} = 1 = x^6\,(x^{-1}wxw^{-1})$ so that, since $[x^2, w] = 1$, we have that $x^{12}[x^2, w] = x^{12} = 1$. By Part 3, this means that $x^6 = 1$ and x^3 is central in G, for any $x \in G$. From Part 4 we have that $[x^2, y] = 1$ for any $y \in G$ since $x^6 = 1$ so that since $x^3 \in Z(G)$, $x \in Z(G)$, that is, G is abelian. This proves the Theorem.

The following result generalizes the identity $x^3 = 1$.

Theorem 6. 3. *Let G satisfy the law*

$$xaybxcydxeyf = 1, \quad \text{for all} \quad x, y \in G, \tag{6. 8}$$

parametrically, for certain fixed $a, b, c, d, e, f \in G$. Then G has exponent 9, satisfies an engel condition of length 2, and $G/Z(G)$ has exponent 3, where $Z(G)$ denotes the center of G.

Proof. We first note the following general result:

If for certain $g, h \in G$ and all $x \in G$, $xgxhx = gh$, then $[g, x, x] = 1$ for all $x \in G$ and $x^9 = 1$. The proof runs as follows: By hypothesis, $xgxhx = x^2gx^2h$ so that $x^2hx = g^{-1}x^{-1}ghx$. Further, $x^2gx^2hx^2 = x^3gx^3hx^3$ so that $gx^2h = xgx^3hx$ and $gx^2h = xgxg^{-1}x^{-1}gxh$. Hence, $gx = xgxg^{-1}xg$, that is, $[g, x, x] = 1$. Further the substitution xg^{-1} for x in $xgxhx = gh$ yields $x^2g^{-1}hx = ghg$, that is, there exists $g_1 \in G$such that $x^2g_1x = g_1$ for all $x \in G$. Thus, $x^4g_1x^2 = g_1$, so that $x^2 = g_1^{-1}x^{-4}g_1$ and $g_1 = x^2g_1x = g_1^{-1}x^{-4}g_1^2x$, or $x^{-1}g_1^{-2}x^4 = g_1^{-2}$. The substitution $x = g_1$ gives $g_1^3 = 1$ so that $x^{-1}g_1x^4 = g_1$. Since $x^8g_1x^4 = g_1$, by hypothesis, this implies finally that $x^9 = 1$ for all $x \in G$.

We now return to the proof of the Theorem. The substitution $y = a^{-1}gb^{-1}$ in (6. 8) for arbitrary (but fixed) $g \in G$ yields $xgxca^{-1}dxea^{-1}gb^{-1}g = 1$ so that, with $x = 1$, $gca^{-1}gb^{-1}dea^{-1}gb^{-1}f = 1$. Thus, $xgxca^{-1}gb^{-1}dx = gca^{-1}gb^{-1}d$, which, by the above remarks, implies that $[g, x, x] = 1$ for any $g \in G$, $x \in G$, and $x^9 = 1$.

Finally, it follows from the above argument that there exists an element $g \in G$ such that $x^2gx = g$, for all $x \in G$. In particular, for xy, where y is arbitrary, $(xy)^2g(xy) = g$, or, $(xy)^2gxg^{-1} \cdot gyg^{-1} = 1$, and since $gxg^{-1} = x^{-2}$, this yields that $(xy)^2x^{-2}y^{-2} = 1$ for any $x, y \in G$. Hence, $y^{-1}xyxyx^{-2}y^{-1} = 1$ so that $[y, x^{-1}][x^{-2}, y^{-1}] = 1$ and since $[x, y, x] = 1$ this implies that $[x, y]^3 = [x^3, y] = 1$, that is, $x^3 \in Z(G)$. This completes the proof of the Theorem. In particular, since $x^2gx = g$, $x^3[x, g^{-1}] = 1$ so that $x \in Z(G)$ implies $x^3 = 1$.

Example. The following group G satisfies the law $x^2gx=g$ for all $x\in G$:

$$G=\{a, b, c, d|a^9=c^9=b^3=d^3=[a, c]=[a, d]=[b, c]=[b, d]=1,$$

$$b^{-1}ab=a^7,\ d^{-1}cd=c^7\}.$$

(Take $g=bd$.) However, there is no element g in $\{a, c\}$, such that $\{a, c\}$ satisfies the same law, because $\{a, c\}$ is abelian of exponent 9. This shows that the class of groups satisfying the law indicated is not closed under forming subgroups.

References

[1] K. W. GRUENBERG, The Engel Elemente of a Soluble Group, *Illinois J. Math.* **3** (1959), 151—168.
[2] W. KAPPF, Die A-Norm einer Gruppe, *Illinois J. Math.* **5** (1961), 187—197.
[3] L. RÉDEI, Die endlichen einstufig nicht nilpotenten Gruppen, *Publ. Math. Debrecen* **4** (1956), 303—324.
[4] E. SCHENKMAN, Group Theory, *Princeton*, 1965.
[5] ZASSENHAUS, The Theory of Groups, *Göttingen*, 1958.
[6] F. W. LEVI, Groups in which the commutator relation satisfies certain algebraic conditions, *J. Indian Math. Soc.*, New Ser. **6** (1942), 87—97
[7] N. D. GUPTA and M. F. NEWMAN, On metabelian groups, *J. Austral. Math. Soc.* **6** (1966), 362—368.

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